Unimodal order in the image of the simplest equivariant chaotic system

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The simplest equivariant chaotic dynamics is investigated in terms of its image, i.e., under the $2\rightarrow1$ mapping allowing one to obtain a projection of the dynamics without any residual symmetry. The inversion symmetry is therefore deleted. The bifurcation diagram can thus be predicted from the unimodal order although the first-return map computed in the original phase space exhibits three critical points. This feature is the same as the one observed on the Burke and Shaw system although this latter system has a rotation symmetry.

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Symmetries have always played an important role in physics, from fundamental formulations of basic principles to concrete applications, and are present in a variety of chaotic systems. Among them, there is the well-known Lorenz system [1] which has a rotation symmetry. This system was the simplest chaotic flow up to 1976 when Rössler [2] deleted all the irrelevant terms from the Lorenz system for obtaining chaotic behavior. The Rössler system no longer has symmetry properties.

During the last five years, some work has been devoted to the search for the simplest set of equations that can generate a chaotic behavior [3]. Most of the time, very simple systems may be written in the form of autonomous third-order differential equations

$$\dddot{x}=\boldsymbol{f}(x,\dot{x},\ddot{x}). \tag{1}$$

Such systems are under study in this paper. Recently, one of us [4] proposed the algebraically simplest example of a dissipative equivariant chaotic system. It consists of three terms including one quadratic nonlinearity:

$$\dddot{x}=-\alpha\ddot{x}+x\dot{x}^2-x. \tag{2}$$

This system can be rewritten as a set of three ordinary differential equations

$$\dot{x}=y, \quad \dot{y}=z, \quad \dot{z}=-\alpha z+xy^2-x, \tag{3}$$

where $y=\dot{x}$ and $z=\ddot{x}$. This system is equivariant, i.e., it obeys the relation $\gamma\cdot\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{f}(\gamma\cdot\boldsymbol{x})$ where γ is a 3×3 matrix defining the symmetry properties. In the present case, the γ matrix

$$\gamma=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix} \tag{4}$$

defines an inversion symmetry $\mathcal{P}$. This means that the vector field $\boldsymbol{f}$ is invariant when (x,y,z) are mapped into $(-x,-y,-z)$. The system (3) has a single fixed point F_0 located at the origin of the phase space. It is a saddle focus with one negative real eigenvalue and two complex conjugate eigenvalues with positive real parts.

Before investigating the dynamical behavior of this equivariant chaotic system, let us give a comment on recent developments in the classification of nonlinear dynamical systems. Thomas and Kaufman [5,6] showed that dynamical systems can be interpreted in terms of feedback full circuits. In particular, the description of dynamical systems by feedback circuits is associated with their fixed points. Such a description could provide a way for classifying dynamical systems directly from their algebraic form. Among the different feedback circuits, those involving all the dynamical variables play a preponderent role in the dynamical behavior since they are the only ones that may be associated with the fixed points of the systems. These circuits are the so-called full circuits [6]. The full circuit of an m-dimensional system may be identified using the m-factor products, the so-called loop products, appearing in the characteristic equation associated with the Jacobian matrix A [6]. In the case of three-dimensional (3D) systems, the determinant consists of six loop products, which are

$$\begin{aligned}\mathrm{Det}(A)=&a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{12}a_{23}a_{31}\\&+a_{13}a_{32}a_{21}-a_{13}a_{31}a_{22}.\end{aligned} \tag{5}$$

Thus, six full circuits may be associated with a 3D dynamical system. In the present case, the Jacobian matrix is

$$A_{ij}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ y^2-1 & 2xy & -\alpha\end{bmatrix}, \tag{6}$$

and consequently, only the loop product $a_{12}a_{23}a_{31}=y^2-1$ is different from zero. Note that, when the chaotic dynamics (1) are considered, only a single full circuit can be obtained. The different types of dynamics will be distinguished by the fixed points and their associated domain where the circuit is either positive or negative. For instance, one negative circuit is a prerequisite for a periodic oscillation and the existence of one positive full circuit a prerequisite for multistationarity.

The circuit associated with the simplest equivariant chaotic dynamics is *ambiguous*, i.e., its sign depends on the location in phase space. When $|y|<1$, the full circuit is negative. It is positive otherwise. The fixed point F_0 is located in the domain where the circuit is negative. Since the full circuit is "pure" in the sense that its loop product does not

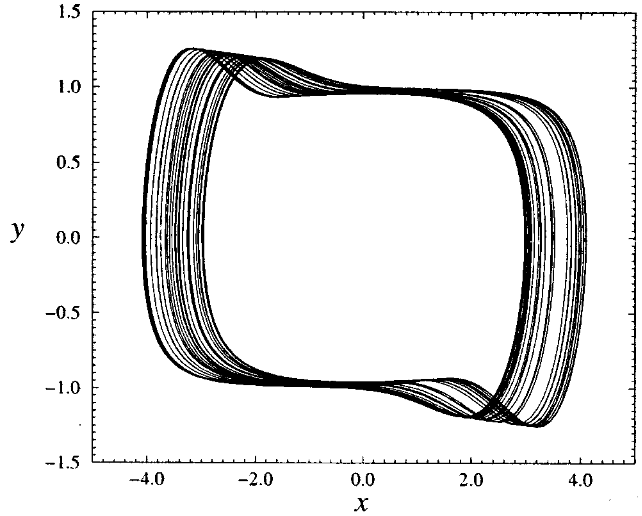

FIG. 1. Chaotic attractor generated by the simplest equivariant chaotic system just before the attractor merging crisis ($\alpha = 2.027\,717$).

contain any on-diagonal element a_{ii}, it should be associated with a fixed point that is a saddle focus. When a pure full circuit is negative (positive), the real eigenvalue is negative (positive) and the real parts of the conjugate complex eigenvalues are positive (negative). In the case of the chaotic system (3), the fixed point F_0 is located in the domain of phase space where the full circuit is negative. This is in agreement with the fact that it is a saddle focus characterized by a negative real eigenvalue. The full circuit is therefore active in the domain $|y|<1$. The other domain is irrelevant for the topology of the dynamics.

When $\alpha = 2.027\,717$, a chaotic attractor (Fig. 1) is obtained when the initial conditions are $(x_0,y_0,z_0) = (4.0,0.0,0.0)$. The symmetry with respect to the origin of the phase space may be easily checked. This attractor is investigated using the Poincaré section

$$P_O = \{(x_n,z_n) \in \mathbb{R}^2 | y_n = 0, \dot{y}_n < 0\}. \tag{7}$$

A bifurcation diagram [Fig. 2(a)] is computed versus α within the interval $[2.027\,717;2.12]$. When α is decreased, two simultaneous period-doubling cascades are observed, one being the symmetric image of the other under the action of γ. After the accumulation point for $\alpha = 2.0840$, the behavior becomes chaotic. Depending on the initial conditions, two attractors, symmetric in phase space, are observed. This feature persists up to an attractor merging crisis. This crisis corresponds to a sudden increase in the size of the chaotic attractor [Fig. 2(a)] and results from two attractors that collide to form a single symmetric attractor [7]. In fact, the crisis appears when each attractor is characterized by a unimodal map for which the symbolic dynamics is complete, i.e., all periodic orbits that may be encoded using two symbols are embedded within the attractor. Within this interval for the α values, the bifurcation diagram can be predicted from the unimodal order [8]. The attractor merging crisis

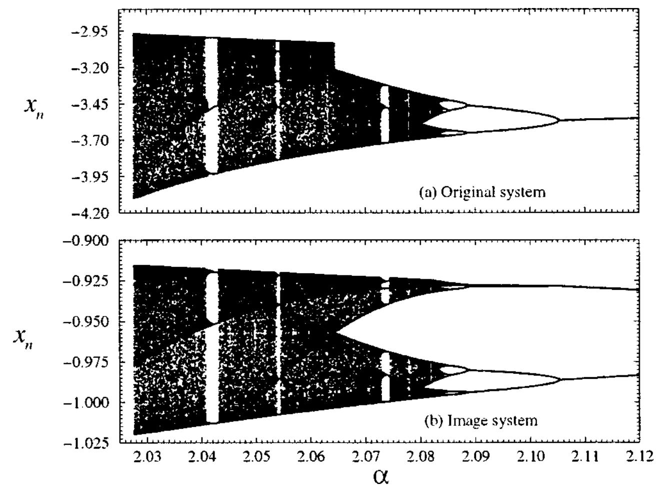

FIG. 2. Bifurcation diagrams versus α for the simplest chaotic dynamics (3) and its image.

occurs for $\alpha \approx 2.0644$. For smaller values of α, a single attractor invariant under the action of γ is observed (as well exemplified in Fig. 1).

When $\alpha < 2.0644$, the single symmetric chaotic attractor is characterized by a multimodal map which may have up to three critical points as observed for $\alpha = 2.027\,717$ (Fig. 3). Periodic orbits are thus encoded on the symbol set $\Sigma_4 = \{\bar{1},0,1,2\}$. In that case, the bifurcation diagram cannot be predicted by the kneading theory. Indeed, when more than one critical point is involved, there is no longer universal order for the creation/destruction of periodic orbits when a control parameter is varied.

Nevertheless, when an equivariant system is considered, it may be possible to simplify the analysis by determining the symmetry properties. Such a procedure was initially introduced by mapping the dynamics in a fundamental domain of the phase space [9] and recently developed by using the image of equivariant systems [10], i.e., the $2 \rightarrow 1$ mapping of the system to obtain a projection of the dynamics without any residual symmetry. The dynamical system (3), which is

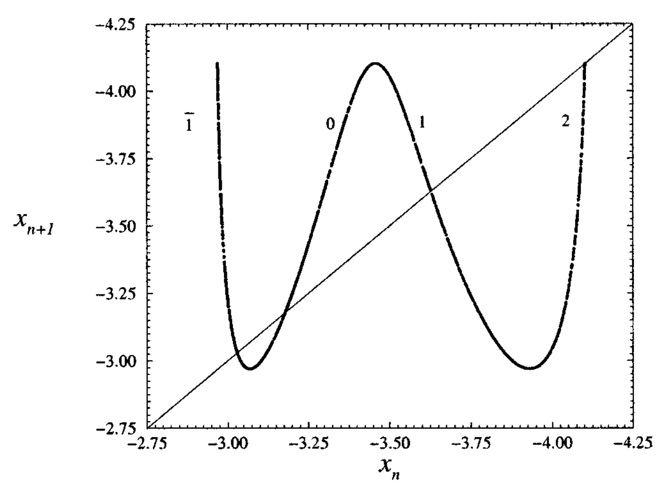

FIG. 3. First-return map of the simplest equivariant chaotic system just before the attractor merging crisis ($\alpha = 2.027\,717$).

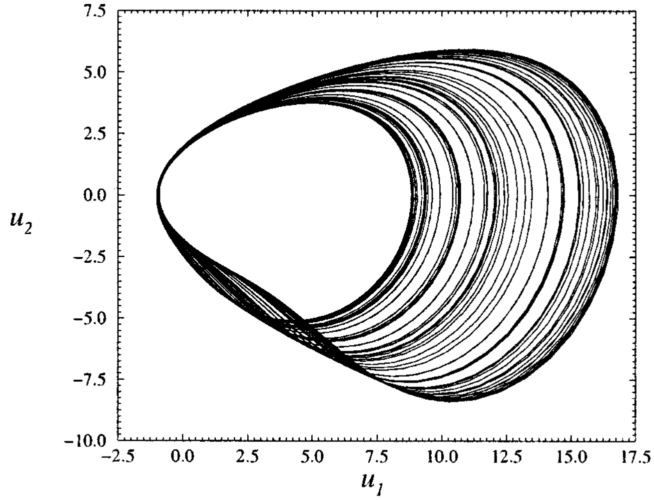

FIG. 4. Image of the simplest equivariant chaotic system just before the attractor merging crisis ($\alpha = 2.027\,717$).

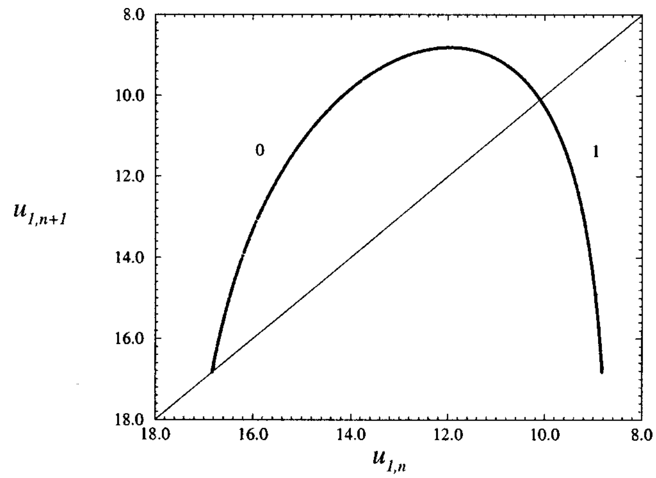

FIG. 5. First-return map of the image system of the simplest equivariant chaotic system just before the attractor merging crisis ($\alpha = 2.027\,717$). A similar map may be obtained using the u_3 variable.

invariant under inversion symmetry, is thus mapped into a locally equivalent dynamical system. This is done by constructing a nonlinear coordinate transformation $(x,y,z) \to (u_1,u_2,u_3)$ in which the coordinates (u_1,u_2,u_3) are invariant under the inversion symmetry $\mathcal{P}$. The elementary polynomials in (x,y,z) of degree up to 2, which are invariant under $\mathcal{P}$, are xy, yz, zx, x^2, y^2, and z^2. The following coordinate transformation is convenient:

$$\varphi = \left| \begin{array}{l} u_1 = x^2 - y^2, \\ u_2 = 2xy, \\ u_3 = z^2. \end{array} \right. \tag{8}$$

The invariant dynamical system equation $\dot{u}_i = g_i(\boldsymbol{u})$ where $\boldsymbol{u} = (u_1,u_2,u_3)$ is determined in a straightforward way:

$$\dot{u}_i = \frac{\partial u_i}{\partial x_j}\frac{\partial x_j}{\partial dt} = \frac{\partial u_i}{\partial x_j} F_j(\boldsymbol{x}) = g_i(\boldsymbol{u}). \tag{9}$$

Using $2x^2 = \rho + u_1$ and $2y^2 = \rho - u_1$ where $\rho = \sqrt{u_1^2 + u_2^2}$, the invariant equations of the chaotic system (3) are

$$\dot{u}_1 = u_2 \pm \sqrt{(\rho - u_1)u_3}, \quad \dot{u}_2 = \rho - u_1 \pm \sqrt{(\rho - u_1)u_3}, \tag{10}$$

$$\dot{u}_3 = -2\alpha u_3 \pm \left[\frac{\rho - u_1}{2} - 1\right]\sqrt{(\rho + u_1)u_3}.$$

The image attractor (Fig. 4) can also be obtained by applying the map φ onto the original attractor. No residual symmetry can be identified.

The image attractor is investigated using the Poincaré section

$$P_I = \{(u_{1,n}, u_{3,n}) \in \mathbb{R}^2 | u_{2,n} = 0, \dot{u}_{2,n} < 0\}. \tag{11}$$

Since the Poincaré section is unidimensional, a first-return map to this Poincaré section may be built with a single variable to define the partition of the attractor. The map is unimodal for $\alpha = 2.027\,717$ (Fig. 5) and not trimodal as observed in the original phase space (Fig. 3). The increasing branch touches the bisecting line. The symbolic dynamics is thus complete. A mapping of a four-branch first-return map into a unimodal map when the symmetry of a system is removed has already been observed in the Burke and Shaw system [9]. The analysis that was performed can be reproduced for the simplest chaotic equivariant system.

When the $2 \to 1$ mapping φ is applied, an orbit embedded within the original attractor is transformed into an orbit embedded within the image attractor. A map Φ from symbolic sequences with p symbols on the set $\Sigma_4 = \{\bar{1},0,1,2\}$ to symbolic sequences of $2p$ symbols on the set $\Sigma_2 = \{0,1\}$ can be defined as [9]

$$\Phi = \left| \begin{array}{l} \Phi(\bar{1}) = 10, \\ \Phi(0) = 11, \\ \Phi(1) = 01, \\ \Phi(2) = 00. \end{array} \right. \tag{12}$$

This transformation Φ maps symbols in blocks of the same parity. For instance, the orbit encoded by $(\overline{1}11)_O$ in the original system is encoded by $\tilde{\Phi}(\overline{1}11)_O = (100101)_I$ in the image system. It may also appear that two different original orbits are mapped into the same image orbit. For instance, this is the case of the original orbits $(\overline{1}11)_O$ and $(2\overline{1}0)_O$ which are both mapped into the image orbit $(100101)_I$. This is the consequence of the $2 \to 1$ mapping which allows us to obtain the image system from the original system. Both periodic orbits appear simultaneously when a control parameter is varied. These connected orbits are therefore characterized by a unique orbit in the image system. Consequently, when the map Φ is inverted and applied to such an orbit, two different symbolic sequences are obtained from the single one in the image system.

TABLE I. Evolution of the orbit spectrum of the simplest equivariant chaotic system in the range [2.027 717,2.12] from the unimodal order. Certain orbits are not present in this table since they are obtained from higher periodic orbits. Let us note the exception of the orbit (1), which is not mapped to an orbit whose period is doubled due to the fact that (1) identifies with (11), which is mapped to (0) by Eq. (13).

$(S)_I$	$(S)_O=\tilde{\Phi}^{-1}(S)_I$			$(S)_I$	$(S)_O\tilde{\Phi}^{-1}(S)_I$		
1		0		10011		$\bar{1}1020$	
10	$\bar{1}$		1	100111	$\bar{1}10$		200
1011	$\bar{1}0$		10	100110	$\bar{1}1\bar{1}$		201
101110	$\bar{1}0\bar{1}$		101	1001	$\bar{1}1$		20
101111	$\bar{1}00$		100	1000	$\bar{1}2$		21
10111		$\bar{1}0010$		100010	$\bar{1}2\bar{1}$		211
10110		$\bar{1}011\bar{1}$		100011	$\bar{1}20$		210
101		$\bar{1}01$		10001		$\bar{1}2021$	
100		112		10000		$\bar{1}2122$	
100101	$\bar{1}11$		$2\bar{1}0$	100001	$\bar{1}21$		220
10010		$\bar{1}112\bar{1}$		100000	$\bar{1}22$		221

Since there is a nonambiguous correspondence between orbits from the original attractor and orbits from the image attractor, the inverted map Φ^{-1} defined as

$$\Phi^{-1}=\left|\begin{array}{l}\Phi^{-1}(10)=\bar{1},\\ \Phi^{-1}(11)=0,\\ \Phi^{-1}(01)=1,\\ \Phi^{-1}(00)=(2),\end{array}\right. \tag{13}$$

allows us to predict the symbolic sequences of the orbits within the original attractor from the orbits embedded within the image attractor. The paired orbits are easily identified by applying a circular permutation toward the left on the symbolic sequence of the original orbit. For instance, from the sequence $(101100)_I$, the sequence $(\bar{1}02)_O$ is obtained. The sequence of the second orbit is obtained as $\Phi^{-1}(011001)_I=(1\bar{1}1)_O$. When a single orbit must be obtained, the transformed sequence under Φ^{-1} is not changed under any circular permutation over the symbols.

Thus, it is possible to predict the evolution of the orbit spectrum associated with the original attractor from the unimodal order identified in the image system (Table I). In particular, the attractor merging crisis between both attractors occurs when the first periodic orbit, with a symbolic sequence containing at least once each symbol $\bar{1}$, 0, and 1, can be constructed. From the bifurcation diagram [Fig. 2(b)] computed for the image system, it may be noted that this attractor merging crisis of the original system appears when the attractor is no longer a period-2 chaotic band, i.e., when the two branches resulting from the period-doubling cascade become a single one ($\alpha \approx 2.0644$). This feature occurs just after the saddle-node bifurcation that induces the two orbits encoded by (101110) and (101111), respectively. In Table I, the reported orbits have a period too small to allow an accurate localization of the attractor merging crisis in the bifurcation diagram.

One of the simplest equivariant chaotic systems has been investigated using its image system. It has been shown that its bifurcation diagram can be described from the unimodal order using a nonambiguous map applied to the symbolic sequences of the periodic orbits embedded within the image attractor. Such a feature was already observed on the Burke and Shaw system, which has a rotation symmetry rather than the inversion symmetry observed on this equivariant chaotic system. This scenario is therefore more general than initially expected. One should note that the images of the Burke and Shaw system and the simplest equivariant chaotic system considered here have the same template, corresponding to a horseshoe template with a global half turn.

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