Ansatz library for global modeling with a structure selection

Claudia S. M. Lainscsek,[1] Christophe Letellier,[2] and Ferdinand Schürrer[3]
[1]Cognitive Science Department, University of California at San Diego, 9500 Gilman Drive, La Jolla, California 92093-0515
[2]Coria UMR 6614, Université de Rouen, Place Emile Blondel, F-76821 Mont Saint-Aignan Cedex, France
[3]Institut für Theoretische Physik, Technische Universität Graz, Petersgasse 16, A-8010 Graz, Austria
(Received 12 May 2000; revised manuscript received 7 February 2001; published 14 June 2001)

The information contained in a scalar time series and its time derivatives is used to obtain a global model for the underlying dynamics. This model provides a description of the time evolution of the system studied in a space spanned by the time series and its successive time derivatives which is expected to be equivalent to the original phase space. Differential models are, in general, very complicated and do not necessarily capture all properties of the original dynamics. The possibility of choosing a form among an ansatz library for the original system which allows a structure selection for the differential model is considered. It allows for the reduction of the complexity of the model and the recovery of the right property when the differential model is transformed back into the space associated with the ansatz.

DOI: 10.1103/PhysRevE.64.016206 PACS number(s): 05.45.−a

I. INTRODUCTION

Experimentally measured data series often have the form of a single variable time series $x(t)$ sampled at regular intervals τ_s. In principle, it is possible to reconstruct phase space topological properties of the underlying dynamics from such a scalar time series [1]. Moreover, models obtained by global modeling [2–4] can be superior to traditional linear modeling methods such as ARMA models. The procedure to find a global dynamical model in the form of a set of coupled ordinary differential equations is often called the ''dynamical inverse problem'' [5,6]. In general, however, one has no prior knowledge of the exact underlying variables of the original system, especially when only a scalar time series is available. In this case, the estimation of a global model can be difficult, since one has no idea of the proper dimensionality or functional model form which may critically affect the estimation procedure. Nevertheless, it is important to use structure selection not only to reduce the number of terms involved in the estimated models but also to correctly select their type, thus improving model quality [7]. It has also been observed that when the right structure is selected, it becomes easier to obtain a model [8].

When a differential model is attempted, all of the information is included in a model function F constituted by a multivariate polynomial depending on the d_E variables, where d_E is the embedding dimension [9]. In such a case, the description of the dynamics is given in a reconstructed space, the so-called differential embedding, spanned by the recorded time series itself and its (d_E-1) derivatives. The number of terms involved in such a model function can be quite large ($\approx$50). Since the obtained differential model may be used to extract analytically some information about the dynamical properties, it is rather important to reduce its length. In order to do that, a structure selection originally introduced in [10] has been recently adapted to the case of differential models [11]. It has been observed that the complexity of the differential model may be reduced and its quality improved, i.e., the parsimonious model generates a dynamical behavior closer to the original one with a simpler model function [12].

Nevertheless, although the differential embedding is expected to be equivalent to the original phase space which is usually unknown, some characteristics of the original dynamics are not always well reproduced when a phase portrait is reconstructed from a single time series. Such a feature is particularly true when the original system has certain symmetry properties. For instance, the Lorenz system [13] generates an attractor which is globally invariant under a rotation around the z axis. When a reconstruction is attempted starting from the x or y variable, the reconstructed attractor possesses an inversion symmetry which is quite different from the original one [14]. It is, therefore, of great interest to apply a technique that would allow a description of the dynamics in a reconstructed phase space with the same symmetry properties even when only a single scalar time series is known.

Our aim is to use an ansatz library for structure selection that allows for the reduction of the complexity of the obtained model and the recovery of some properties of the original dynamics. The ansatz corresponds to a chosen structure for the original system generating the recorded time series. Using an embedding dimension equal to 3, we choose an ansatz with respect to the coordinates (x,y,z) which allows the inversion of the map Φ: $(x,y,z)\rightarrow(X,Y,Z)$, where (X,Y,Z) are the time series itself and its successive derivatives, respectively. The model functions $\tilde{F}_i$ referring to the recorded data have a structure corresponding to the ansatz A_i from the ansatz library. The adequate ansatz is identified when a transformation back into the ansatz coordinates (x,y,z) is possible. The applicability of such a procedure is exemplified by using a library constituted by two different ansatz and two different data sets.

The paper is organized as follows. In Sec. II the global modeling technique is presented as well as how to derive an ansatz library for structure selection. This method works in two steps: (i) estimating the function F with each ansatz of the library and (ii) trying to transform the model back into a model with the ansatz structure. The adequate ansatz is the one for which the transformation is possible. In Sec. III this technique is applied to two dynamical systems, namely the Lorenz and the Rössler systems using two ansatz from the library. The obtained models are then compared to the original ones. Section IV is the conclusion.

II. GLOBAL MODELING TECHNIQUE

An approach for global modeling with structure selection is introduced. The first step is to build an ansatz library. Each ansatz represents a predefined structure for the differential model to estimate. Indeed, starting from a given ansatz, the second step to define the one and only restricted structure for the differential model can be computed. Thus, from the ansatz library a differential model library is computed. The second step consists of performing global modeling as usual [3,15] with each differential model. The third step is to select the ansatz corresponding to an invertible map that allows us to express the ansatz coefficients versus the model coefficients. Indeed, we assume that when this map is invertible, the ansatz is close to the original system. The ansatz model is thus used to perform a better analysis of the dynamics. This procedure will be detailed in the following way. The usual global modeling technique to obtain differential models is reviewed in Sec. II A Section II B is devoted to building the ansatz library, and Sec. II C explains how the structure of the differential model is selected from the predefined ansatz, that is, how to map the ansatz to the model in a very simple case. The general case is developed in Sec. II D.

A. General approach

The modeling method presented in this paper is applied to systems whose dimension D of the original phase space is 3. We limit ourselves to the cases where the embedding dimension d_E is also fixed to be equal to 3. Our method is based on the global vector field modeling technique introduced by Gouesbet and co-workers [3,9]. Very often the time evolution of all dynamical variables required for a complete description of the system studied is not known, and we cannot recover the original set of differential equations. We usually know a single scalar time series, and only a global model is obtained by using a reconstructed space spanned by coordinates derived from the recorded time series [16].

Let us consider a dynamical system in $\mathbb{R}^3(u,v,w)$,

$$g \equiv \begin{cases} \dot{u} = f_1(u,v,w) \\ \dot{v} = f_2(u,v,w) \\ \dot{w} = f_3(u,v,w) \end{cases}. \tag{1}$$

Starting from a single time series $\{u_i\}_{i=1}^N$, where N is the number of points and $u_i = u(t_i)$, we obtain a set of independent variables by computing the derivatives

$$X_i = u_i\,,$$

$$Y_i = \frac{du_i}{dt}, \tag{2}$$

$$Z_i = \frac{d^2u_i}{dt^2}.$$

A representation of the phase portrait is then given in a differential embedding $\mathbb{R}^3(X,Y,Z)$. With such a set of coordinates, a modeling procedure is applied to obtain a model from the recorded time series $\{u_i\}_{i=1}^N$. Such a model reads

$$G \equiv \begin{cases} \dot{X} = Y \\ \dot{Y} = Z \\ \dot{Z} = F(X,Y,Z,\alpha_n) = \sum_{n=1}^{N_\alpha} \alpha_n P_n \end{cases}, \tag{3}$$

where α_n are the coefficients of the model function F to be estimated and P_n are the monomials $X^iY^jZ^k$ [3]. The indices (i,j,k) for monomials may also be negative yielding a rational model. System (3) is called the *differential model* and its parameters can be obtained by solving an overdetermined system of N equations with N_α unknown coefficients $\alpha_n (N \geqslant N_\alpha)$ reading as

$$\begin{pmatrix} \dot{Z}_1 \\ \dot{Z}_2 \\ \vdots \\ \dot{Z}_N \end{pmatrix} = \begin{pmatrix} F(X_1,Y_1,Z_1) \\ F(X_2,Y_2,Z_2) \\ \vdots \\ F(X_N,Y_N,Z_N) \end{pmatrix}. \tag{4}$$

In the method discussed in [3], this model function $F(X,Y,Z,\alpha_n)$ is obtained by using a Fourier expansion on the basis of orthonormal multivariate polynomials generated from the data set. A SVD procedure may also be preferred to solve this set of equations [15]. The latter is used in this work.

B. Ansatz library

When a differential model is attempted, the model function $F(X,Y,Z)$ is usually estimated by using a polynomial expansion that involves a larger set of terms than required. The spurious terms increase the complexity of the model and decrease its qualitity, i.e., the asymptotic behavior generated by the obtained model may differ more from the original dynamics. In order to avoid that, structure selection must be performed. Different techniques exist. One of them starts with a model function estimated on a full polynomial expansion up to a certain order and deletes spurious terms [12]. In this paper we propose a quite different procedure. Rather than modifying or building a model structure for a particular case, we start with different complete model structure, i.e., the so-called ansatz library, out of which we choose the best. Each ansatz is thus used for selecting the structure of the model function, i.e., for reducing the number of coefficients involved in the estimated function.

In order to obtain a model with an adequate structure that expresses the model equations in a form closer to the original one than the differential model (3), an ansatz library is used to select a structure for the original system. The ansatz are based on dynamical variables (x,y,z) which may differ from the unknown original dynamical variables (u,v,w). The ansatz are selected in order to be able to invert the coordinate transformation Φ between the dynamical variables (x,y,z) on which the ansatz is built and the derivative coordinates (X,Y,Z) induced by the recorded time series. In the subsequent part of this paper, the x variable will designate the observable in order to avoid redundancy, i.e., the system investigated will be rewritten in the form that the observable corresponds to the first ordinary differential equation.

Let us limit ourselves to the quite general dynamical system equations

$$
\begin{aligned}
\dot{x} &= a_0 + a_1 x + a_2 y + a_3 z + a_4 x^2 + a_5 xy \\
&\quad + a_6 xz + a_7 y^2 + a_8 yz + a_9 z^2 \\
&= f_1(x,y,z), \\
\dot{y} &= b_0 + b_1 x + b_2 y + b_3 z + b_4 x^2 + b_5 xy \\
&\quad + b_6 xz + b_7 y^2 + b_8 yz + b_9 z^2 \\
&= f_2(x,y,z), \\
\dot{z} &= c_0 + c_1 x + c_2 y + c_3 z + c_4 x^2 + c_5 xy \\
&\quad + c_6 xz + c_7 y^2 + c_8 yz + c_9 z^2 \\
&= f_3(x,y,z),
\end{aligned}
\tag{5}
$$

where the right-hand side is constituted by order two multivariate polynomials. There is no conceptual difficulties to extend this approach to higher-order polynomial systems. This dynamical system is associated with the phase space spanned by the dynamical variables (x,y,z). By using the derivative coordinates, a phase space may be reconstructed and a differential model (3) can be obtained. Our objective is to use some ansatz, which are a subpart of the general form (5) to select the structure of the differential model estimated from a time series. Since we do not know *a priori* the exact model form when we are faced with an experimental system, we can only assume that it could correspond to a given ansatz defined by a set of nonzero coefficients $\{a_i, b_j, c_k\}$ among those of the general form (5). Starting from a given ansatz A, the derivative coordinates can be expressed versus the dynamical variables involved in A according to the transformation

$$
\Phi_p = \begin{cases} X = x \\ Y = f_1(x,y,z) \\ Z = \dfrac{\partial f_1}{\partial x} f_1 + \dfrac{\partial f_1}{\partial y} f_2 + \dfrac{\partial f_1}{\partial z} f_3 . \end{cases}
\tag{6}
$$

This function must be invertible to express the ansatz coordinates (x,y,z) versus the derivative coordinates (X,Y,Z). Such an inverse map Φ_p^{-1} is required to express the function $F(X,Y,Z)$ versus the ansatz variables. Most of the possibilities should correspond to an invertible map Φ_p constituted by polynomial functions. The map Φ_p is necessarily invertible when it is restricted to be on the form

$$
\Phi_p = \begin{cases} X = x \\ Y = \phi_1(x,y) \\ Z = \phi_2(x,y,z) \end{cases} ,
\tag{7}
$$

where $\phi_1(x,y)$ and $\phi_2(x,y,z)$ are polynomial functions. Since we would like to avoid terms with noninteger power like $\sqrt{x}$ or other, the function $\phi_1(x,y)$ must be linear in y, i.e., can be decomposed into two functions

$$
\phi_1(x,y) = h_1(x) + g_1(x) y,
\tag{8}
$$

where $h_1(x)$ may be a second-order polynomial in x. Since we limited ourselves to functions f_i's constituting the ansatz A as second-order polynomials, $g_1(x)$ must be a first-order polynomial in x and may have only two different forms,

$$
\begin{aligned}
g_1(x) &= \eta_1, \\
g_1(x) &= \eta_1 x,
\end{aligned}
\tag{9}
$$

where η_1 is a real coefficient. Starting from Eq. (8) and using $g_1(x) = \eta_1 x^n$ with $n \in \mathbb{N}_0; \leqslant 1$, we obtain

$$
\begin{aligned}
\phi_2(x,y,z) &= f_2 g_1(x) + y^2 g_1(x) g_1'(x) \\
&\quad + y h_1(x) g_1'(x) + y g_1(x) h_1'(x) \\
&\quad + h_1(x) h_1'(x) \\
&= x^n f_2 \eta_1 + n x^{2n-1} y^2 \eta_1{}^2 \\
&\quad + n x^{n-1} y \eta_1 h_1(x) + x^n y \eta_1 h_1'(x) \\
&\quad + h_1(x) h_1'(x).
\end{aligned}
\tag{10}
$$

Using similar arguments for $\phi_1(x,y)$, the function f_2 must be linear in z, i.e.,

$$
f_2(x,y,z) = h_2(x,y) + g_2(x,y) z.
\tag{11}
$$

The function ϕ_2 may thus be written in the form

$$
\begin{aligned}
\phi_2(x,y,z) &= n x^{2n-1} y^2 \eta_1{}^2 + x^n z \eta_1 g_2(x,y) \\
&\quad + n x^{n-1} y \eta_1 h_1(x) + x^n \eta_1 h_2(x,y) \\
&\quad + x^n y \eta_1 h_1'(x) + h_1(x) h_1'(x).
\end{aligned}
\tag{12}
$$

When $g_1(x) = \eta_1$, we have two possibilities for $g_2(x,y)$, which are

$$
\begin{aligned}
g_2(x,y) &= \eta_2. \\
g_2(x,y) &= \eta_2 x,
\end{aligned}
\tag{13}
$$

and when $g_1(x) = \eta_1 x$, only $g_2(x,y) = \eta_2$ is suitable. When $h_1(x) = 0$, the funcitons ϕ_1 and ϕ_2 reduce to

$$
\begin{aligned}
&\phi_1 = x^n y \eta_1, \\
&\phi_2 = n x^{2n-1} y^2 \eta_1^2 + x^n z \eta_1 g_2(x,y) + n^n \eta_1 h_2(x,y).
\end{aligned}
\tag{14}
$$

In this case the function $g_2(x,y)$ can take the additional form

$$g_2(x,y)\,\eta_2 y. \tag{15}$$

These three possibilities for $g_2(x,y)$ can be summarized as

$$g_2(x,y)=\eta_2 x^m y^p z,$$

with

$$\begin{aligned} &h_1(x)=0: \quad m,p\in\mathbb{N}_0; \quad n+m\leqslant 1,\\ &h_1\neq 0: \quad p=0;n,m\in\mathbb{N}_0; \quad n,m\leqslant 1. \end{aligned} \tag{16}$$

The y and z variables are thus given as

$$\begin{aligned} y&=\frac{Y-h_1(x)}{x^n\eta_1},\\ z&=\frac{Z}{x^n\eta_1 g_2(x,y)}-\frac{nx^{n-1}y^2\eta_1}{g_2(x,y)}-\frac{nyh_1(x)}{xg_2(x,y)}\\ &\quad-\frac{h_2(x,y)-yh_1'(x)}{g_2(x,y)}-\frac{h1(x)h_1'(x)}{x^n\eta_1 g_2(x,y)}\\ &=\frac{Z}{x^{m+n}\eta_1\eta_2}-\frac{nx^{n-1-m}y^{2^{\eta_1}}}{\eta_2}\frac{nyh_1(x)}{x^{m+1}\eta_2}\\ &\quad-\frac{h_2(x,y)}{x^m\eta_2}-\frac{yh_1'(x)}{x^m\eta_2}-\frac{h_1(x)h_1'(x)}{x^{m+n}\eta_1\eta_2} \end{aligned} \tag{17}$$

and for $h_1(x)=0$ they are

$$\begin{aligned} y&=\frac{Y}{x^n\eta_1},\\ z&=\frac{Z}{x^{m+n}y^P\eta_1\eta_2}-\frac{nx^{n-m-1}y^{2-p}\eta_1}{\eta_2}-\frac{h_2(x,y)}{x^m y^p\eta_2}. \end{aligned} \tag{18}$$

We therefore have four different possibilities for the funciton $f_1(x,y)=\eta_0 h_1(x)+\eta_1 x^n y$ which are

$$\begin{array}{lcl} \eta_0=1,\ n=0,\ \eta_1=a_2 &:& f_1(x,y) = h_1(x)+\boxed{a_2 y}\\ \eta_0=1,\ n=1,\ \eta_1=a_5 &:& f_1(x,y) = h_1(x)+\boxed{a_5 xy}\\ \eta_0=0,\ n=0,\ \eta_1=a_2 &:& f_1(x,y) = \boxed{a_2 y}\\ \eta_0=0,\ n=1,\ \eta_1=a_5 &:& f_1(x,y) = \boxed{a_5 xy} \end{array} \tag{19}$$

with

$$h_1(x)=a_0+a_1x+a_4x^2$$

and three possibilities fot function $f_2(x,y,z)=h_2(x,y)+\eta_2 x^m y^p z$, which are

$$\begin{array}{lcl} m=0,\ p=0,\ \eta_2=b_3 &:& f_2(x,y) = h_2(x,y)+\boxed{b_3 z}\\ m=1,\ p=0,\ \eta_2=b_6 &:& f_2(x,y) = h_2(x,y)+\boxed{b_6 xz}\\ m=0,\ p=1,\ \eta_2=b_8 &:& f_2(x,y) = h_2(x,y)+\boxed{b_8 yz} \end{array} \tag{20}$$

with

$$h_2(x,y)=b_0+b_1x+b_2y+b_4x^2+b_5xy+b_7y^2.$$

When these functions are combined with the function f_3 of the general ansatz (5), four different ansatz are obtained,

$$A_1\equiv\begin{cases}\dot{x}=a_0+a_1x+\boxed{a_2y}+a_4x^2\\ \dot{y}=b_0+b_1x+b_2y+b_4x^2+b_5xy+\boxed{b_6xz}+b_7y^2\\ \dot{z}=c_0+c_1x+c_2y+c_3z+c_4x^2+c_5xy+c_6xz+c_7y^2+c_8yz+c_9z^2\end{cases}, \tag{21}$$

$$A_2\equiv\begin{cases}\dot{x}=a_0+a_1x+\boxed{a_2y}+a_4x^2\\ \dot{y}=b_0+b_1x+b_2y+\boxed{b_3z}+b_4x^2+b_5xy+b_7y^2\\ \dot{z}=c_0+c_1x+c_2y+c_3z+c_4x^2+c_5xy+c_6xz+c_7y^2+c_8yz+c_9z^2\end{cases}, \tag{22}$$

$$A_3\equiv\begin{cases}\dot{x}=a_0+a_1x+a_4x^2+\boxed{a_5xy}\\ \dot{y}=b_0+b_1x+b_2y+b_4x^2+b_5xy+\boxed{b_6xz}+b_7y^2\\ \dot{z}=c_0+c_1x+c_2y+c_3z+c_4x^2+c_5xy+c_6xz+c_7y^2+c_8yz+c_9z^2\end{cases}, \tag{23}$$

$$A_4\equiv\begin{cases}\dot{x}=a_0+a_1x+a_4x^2+\boxed{a_5xy}\\ \dot{y}=b_0+b_1x+b_2y+\boxed{b_3z}+b_4x^2+b_5xy+b_7y^2\\ \dot{z}=c_0+c_1x+c_2y+c_3z+c_4x^2+c_5xy+c_6xz+c_7y^2+c_8yz+c_9z^2\end{cases}. \tag{24}$$

Note that the leading coefficients of the framed terms must be nonzero otherwise the map Φ_p cannot be inverted. Two other ansatz are remaining. They correspond to the case where $g_2(x,y)$ is equal to μy and read as

$$
A_5 \equiv \begin{cases} \dot{x} = \boxed{a_2 y} \\ \dot{y} = b_0 + b_1 x + b_2 y + b_4 x^2 + b_5 xy + b_7 y^2 + \boxed{b_8 yz} \\ \dot{z} = c_0 + c_1 x + c_2 y + c_3 z + c_4 x^2 + c_5 xy + c_6 xz + c_7 y^2 + c_8 yz + c_9 z^2 \end{cases}
$$

$$
A_6 \equiv \begin{cases} \dot{x} = \boxed{a_5 xy} \\ \dot{y} = b_0 + b_1 x + b_2 y + b_4 x^2 + b_5 xy + b_7 y^2 + \boxed{b_8 yz} \\ \dot{z} = c_0 + c_1 x + c_2 y + c_3 z + c_4 x^2 + c_5 xy + c_6 xz + c_7 y^2 + c_8 yz + c_9 z^2 \end{cases} \tag{25}
$$

The last two ansatz are less interesting than the first four because the functions f_1 are very constrained.

This procedure may be extended straightforward to the case of higher-order polynomial functions f_i's as well as for higher-dimensional systems. It may appear that other ansatz could exist but no systematic way for generating them has been obtained so far. We conjecture that they are very rare.

C. Differential model estimation

When a model is attempted from a scalar time series, only a phase portrait reconstructed using the derivative coordinates can be obtained. The model is thus obtained by estimating the function $F(X,Y,Z)$ of a model of form (3). Inorder to avoid numerous terms in the estimated model functions $\tilde{F}(X,Y,Z)$, its structure is selected to correspond to an ansatz A_i. For the sake of clarity, let us start with a simple case.

The Rössler system [17], one of the most simple systems that produce chaos, reads

$$
\begin{aligned} \dot{u} &= -v - w, \\ \dot{v} &= u + av, \\ \dot{w} &= b - \mathcal{C}w + uw, \end{aligned} \tag{26}
$$

where $(a,b,\mathcal{C}) = (0.2,0.2,35.0)$ are the control parameters. When the v variable of the Rössler system is chosen as the observable $(x=v)$, the Rössler system belongs to ansatz A_2 when the coordinate transformation $(x,y,z)=(v,u,w)$ is used. The ansatz A_2 is thus reduced to

$$
A_0 \equiv \begin{cases} \dot{x} = a_1 x + \boxed{a_2 y} \\ \dot{y} = b_1 x + \boxed{b_3 z} \\ \dot{z} = c_0 + c_3 z + c_8 yz \ , \end{cases} \tag{27}
$$

where

$$
\begin{aligned} & a_1 = a, \quad a_2 = 1.0, \\ & b_1 = -1.0, \quad b_3 = -1.0, \\ & c_0 = b, \quad c_3 = -\mathcal{C}, \quad c_8 = 1.0. \end{aligned} \tag{28}
$$

The other coefficients involved in ansatz A_2 are set to zero. This reduced ansatz will be designated as the ansatz A_0. It will be used only for introducing the method. In this case, the coordinate transformation Φ_0 given in Eq. (7) reads

$$
\Phi_0 = \begin{cases} X = x \\ Y = a_1 x + a_2 y \\ Z = (a_1^2 + a_2 b_1)x + a_1 a_2 y + a_2 b_3 z \end{cases} \tag{29}
$$

and its inverse

$$
\Phi_0^{-1} = \begin{cases} x = X \\ y = \dfrac{Y - a_1 X}{a_2} \\ z = \dfrac{Z - a_2 b_1 X - a_1 Y}{a_2 b_3}. \end{cases} \tag{30}
$$

It is clear in this example that the leading coefficients a_2 and b_3 must be nonzero to define the inverse map Φ_0^{-1}. Starting from the reduced ansatz (27), it is possible to determine the exact model function $F_0(X,Y,Z)$. The exact model function is directly obtained by computing the derivative of the function $\phi_2(x,y,z)$ of Eq. (7), which is thus expressed versus the ansatz coordinate (x,y,z),

$$
\begin{aligned} F_0(x,y,z) = {} & a_2 b_3 c_0 + a_1(a_1^2 + 2a_2 b_1)x + a_2(a_1^2 + a_2 b_1)y \\ & + a_2 b_3(a_1 + c_3)z + a_2 b_3 c_8 yz. \end{aligned} \tag{31}
$$

It remains to transform the original coordinates (x,y,z) by using the inverted map Φ_0^{-1} to obtain the model function

$$
\begin{aligned} F_0(X,Y,Z) = {} & \alpha_1 + \alpha_2 X + \alpha_3 X^2 + \alpha_4 Y \\ & + \alpha_5 XY + \alpha_6 Y^2 + \alpha_7 Z + \alpha_8 XZ + \alpha_9 YZ, \end{aligned} \tag{32}
$$

where

TABLE I. Coefficients α_n of the model function F expressed versus the original coefficients $\{a_i,b_j,c_k\}$. The exact values $\alpha_{n,m}$ and the estimated values $\tilde{\alpha}_n$ and $\tilde{\alpha}_m$ are reported for ansatz A_0 and A_2, respectively. In the case of ansatz A_2, considered later, all of the coefficients not reported here are zero except α_3, α_{28}, and α_{35} which are equal to 0.001.

n	m	$P_{n,m}$	α_n	$\alpha_{n,m}$	$\tilde{\alpha}_n$ for A_0	$\tilde{\alpha}_m$ for A_2
1	1	1	$a_2b_3c_0$	−0.20	−0.1634	0.02
2	2	X	$-a_2b_1c_3$	−35.00	−34.9880	−35.336
3	3	X^2	$a_1b_1c_8$	−0.20	−0.2001	−0.207
4	11	Y	$a_2b_1-a_1c_3$	6.00	5.9972	6.067
5	12	XY	$\frac{a_1^2c_8}{a_2}-b_1c_8$	1.04	1.0397	1.064
6	18	Y^2	$-\frac{a_1c_8}{a_2}$	−0.20	−0.2000	−0.205
7	27	Z	a_1+c_3	−34.80	−34.7877	−35.119
8	28	XZ	$-\frac{a_1c_8}{a_2}$	−0.20	−0.2000	−0.206
9	32	YZ	$\frac{c_8}{a_2}$	1.00	0.9996	1.023

$$\varphi_0 \equiv \begin{cases} \alpha_1 = a_2b_3c_0 \\ \alpha_2 = -a_2b_1c_3 \\ \alpha_3 = a_1b_1c_8 \\ \alpha_4 = a_2b_1-a_1c_3 \\ \alpha_5 = \dfrac{a_1{}^2c_8}{a_2}-b_1c_8 \\ \alpha_6 = -\dfrac{a_1c_8}{a_2} \\ \alpha_7 = a_1+c_3 \\ \alpha_8 = -\dfrac{a_1c_8}{a_2} \\ \alpha_9 = \dfrac{c_8}{a_2} \end{cases}. \tag{33}$$

The inverse of this map φ_0 will allow us to derive the ansatz model from the differential model.

The α_n's are the coefficients which are to be estimated when a differential model is attempted using the global modeling technique described in the preceding section. In the present case, a differential model has been estimated by using a time series numerically recorded at a sampling rate equal to 100 Hz and constituted of 3000 data points. All the data points are taken from a time series of the v variable of the Rössler system (26). For each data point retained, the derivatives have been estimated by using an interpolation polynomial. These interpolation polynomials are centered at each point by using the nearest neighbors. Derivatives are obtained afterward by taking analytically derivatives of these polynomials. Then, a vector constituted by the v variable and its three successive derivatives is retained regularly by 0.1 s. The system $\dot{Z}=\tilde{F}_0(X,Y,Z)$ is solved by using a SVD algorithm. The estimated coefficients $\tilde{\alpha}_n$'s are reported in Table I.

The model obtained is quite close to the expected one, i.e., the estimated function $\tilde{F}_0(X,Y,Z)$ is constituted of terms that have estimated coefficients $\tilde{\alpha}_n$ very close to the exact values α_n's. In comparison to the general approach used in [3], the ansatz approach allows us to reduce the number of terms on which the function $F_0(X,Y,Z)$ is estimated. Consequently, it allows us a so-called *structure selection* for the differential model. This is the first interesting point of the ansatz approach. Nevertheless, the plane projections of the differential model [Figs. 1(d), 1(e), and 1(f)] have nothing to do with the plane projections of the original Rössler system under the form of ansatz A_0 [Figs. 1(a), 1(b), and 1(c)]. Such a feature results from the fact that the derivative coordinates are coupled in a very different way than the ansatz variables (x,y,z) are. However, since we assume an ansatz for the structure of the original system, an ansatz model can be obtained using the inverse map φ_0^{-1}.

D. Ansatz model

Let us continue with our simple example. Equation (33) defines a map φ_0 between the ansatz coefficients $\{a_i,b_j,c_k\}$ and the estimated model coefficients $\tilde{\alpha}_n$'s. If this map can be inverted, the ansatz coefficients can be deduced and the differential model can be transformed into an ansatz model having the same form as the ansatz A_0. Because only seven coefficients are present in the ansatz A_0 and nine coefficients $\tilde{\alpha}_n$ in the estimated model function $\tilde{F}_0(X,Y,Z)$, the problem is not fully determined. Three ansatz coefficients, a_2, b_1, and c_8, are not analytically determined since they are only involved in two linear independent equations. Nevertheless, due to a small departure in numerical values of the estimated

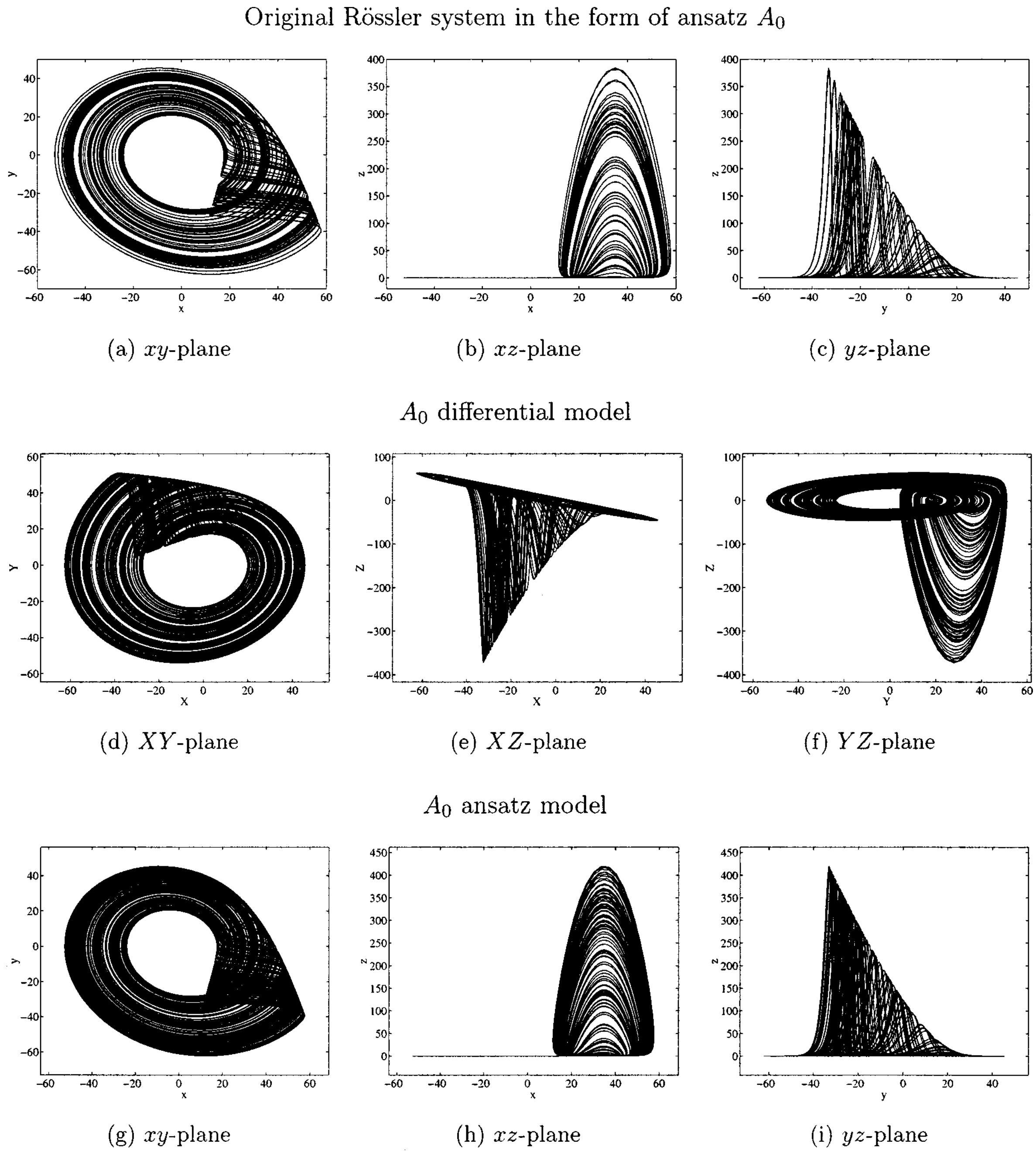


FIG. 1. Plane projections of the attractor generated by the Rössler system in the form of ansatz A_0 with $(a,b,\mathcal{C})=(0.2,0.2,35.0)$. Plane projections of the corresponding differential model and the ansatz model are also shown.

model coefficients $\tilde{\alpha}_n$'s, we have, in fact, six numerically independent equations for these three coefficients which can be obtained. Also, the ansatz coefficients b_3 and c_0 are only involved in the single model coefficient α_1. Consequently, the problem is undetermined for these two ansatz coefficients. When the map φ_0 is inverted using a Gauss-Newton method with a cubic quadratic line search procedure as implemented in Matlab, the two ansatz coefficients b_3 and c_0 may vary significantly from one differential model to the other. The coefficients of the best differential model are reported in Table II. An error up to 70% is found for the ansatz coefficient b_3. The errors will induce a significant scaling in the variables without any other change in the dynamics. Indeed, the value of the control parameter c_0 only affects the scale of the dynamics [20]. Thus, the attractor generated by this ansatz model is topologically equivalent to the ansatz system [Figs. 1(g), 1(h), and 1(i)]. This ansatz model has the great advantage of presenting very similar plane projections like that for the ansatz system, i.e., the couplings between the dynamical variables involved in the ansatz model are very similar to those present in the ansatz system (27).

When more general ansatz are used, it may happen that two differential models are successfully obtained from a given time series. In this case, the structure selection deletes spurious terms and adds missing ones but the model functions are not sufficiently restricted to allow a clear identifi-

TABLE II. Estimated coefficients $\{a_i, b_j, c_k\}$ for the original Rössler system and for the ansatz model after inverting the map φ_0. A significant error is found for the coefficients b_3 and c_0 since they remain underdetermined. Nevertheless, these errors mainly affect the scaling of the variable but not the topology of the attractor.

	a_1	a_2	b_1	b_3	c_0	c_3	c_8
Exact	0.20	1.00	−1.00	−1.00	0.2	−35.0	1.00
Estimated	0.20	1.00	−1.00	−1.70	0.1	−35.0	1.00
Relative error	0.0	0.0	0.0	70%	50%	0.0	0.0

cation of the best ansatz. This can be done when the differential model is transformed back into a model built with the same variables like those involved in the ansatz. The $\{a_i, b_j, c_k\}$ coefficients are thus expressed by inverting the map φ using a Gauss-Newton method with a cubic quadratic line search procedure as implemented in Matlab. If an inversion is possible for one of the two ansatz, one may conclude that the right ansatz is selected. It seems that there is a very low probability for having two differential models built from different ansatz which may be transformed back. This ansatz identification will be exemplified in the next section.

To summarize, the chain of computations is as follows:

(1) Compute the coefficients $\alpha_p^{A_n}$ from a given time series for all ansatz A_p of the ansatz library by using SVD and differential models. For the next computation step, the values $\alpha_p^{A_n}$ for each ansatz A_p are used.

(2) In order to check which ansatz is adequate, try to invert each map φ_p for computing the ansatz coefficients $\{a_i, b_j, c_k\}$. Only ansatz which are appropriate to the time series used allow us to solve the inverse transformation φ_p.

The relationships between the different spaces (i.e., the original phase space, the ansatz space, the differential embedding, and the space associated with the differential model) used in this paper are summarized in Fig. 2.

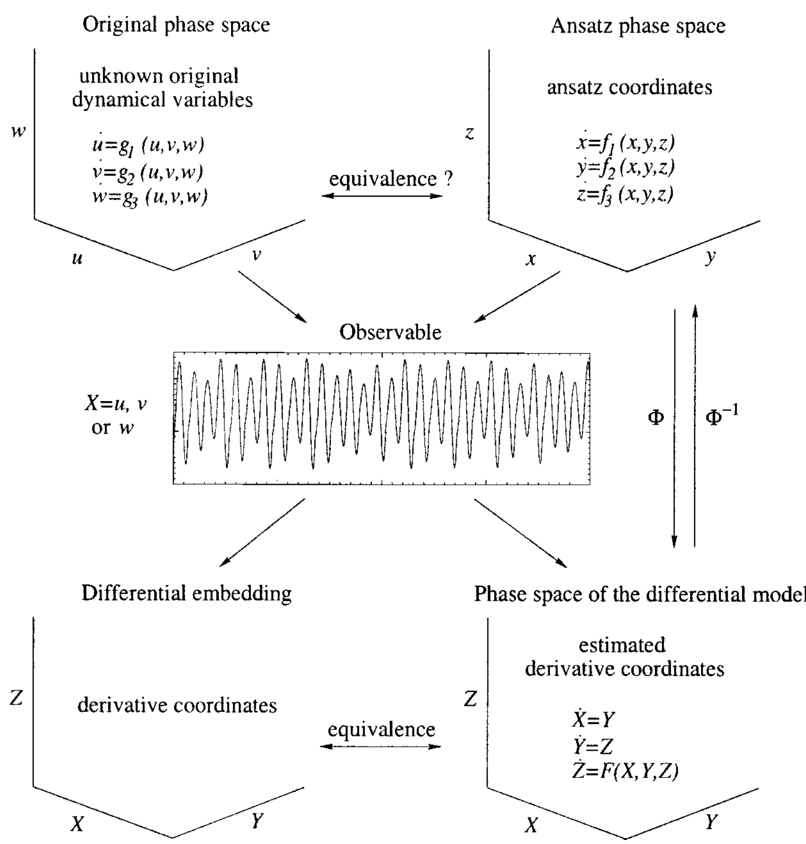
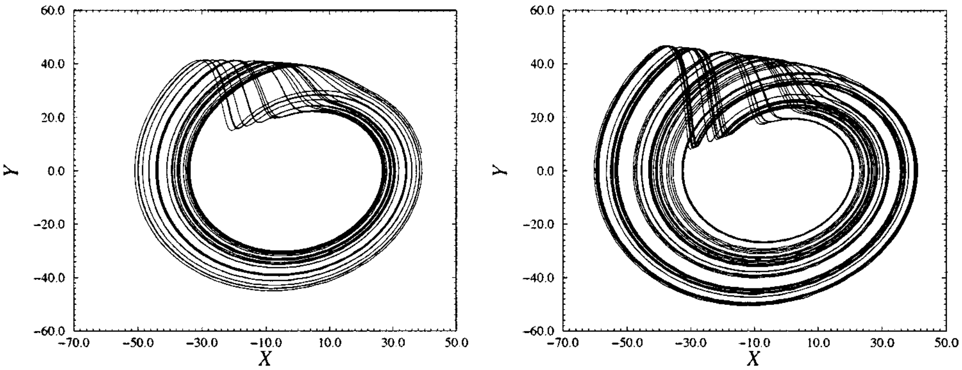


FIG. 2. The original phase space is unknown. Only a scalar time series is recorded. Usually, the observable is considered to be one of the dynamical variables (u,v,w) from the original phase space. Starting with this observable, the derivative coordinates are computed to obtain a differential embedding. This is the single representation of the attractor that can be directly extracted from the data. Thus a differential model is attempted using the derivative coordinates (X,Y,Z). The integration of the estimated differential model induces a phase space spanned by the model derivative coordinates (X,Y,Z). In this paper these coordinates are not distinguished from the derivative coordinates directly computed from the experimental time series. In order to impose a structure selection on the model function $F(X,Y,Z)$, an ansatz is assumed to match with the dynamics underlying the original dynamical variables (u,v,w). The coordinate transformation Φ: $(x,y,z) \rightarrow (X,Y,Z)$ between the ansatz phase space and the phase space of the differential model must be invertible to allow the transformation of the differential model into a model in the ansatz form. But numerically, it is in fact the map φ: $(\alpha_n) \rightarrow (a_i, b_j, c_k)$ which is inverted. This is possible until the leading coefficients are not equal to zero. When this is observed, one may expect that the ansatz phase space and the original phase space are very close.



(a) A_1 differential model (b) A_2 differential model

FIG. 3. Chaotic attractors generated by differential models estimated with ansatz A_1 and A_2 when the v variable of the Rössler system is the observable.

III. NUMERICAL APPLICATIONS

A. Rössler system

Two quite different original systems are used as test cases. First, we consider the Rössler system (26). When the observable is the u variable, the Rössler system does not belong to any ansatz identified for second-order polynomial functions. Thus, following the procedure previously developed, no invertible coordinate transformation Φ between ansatz coordinates (x,y,z) and derivative coordinates ($X=x$, $Y=\dot{x}$, $Z=\ddot{x}$) can be obtained. The case of the w variable is a little bit different. When the ansatz coordinates are $(x,y,z)=(w,v,u)$, the Rössler system may be rewritten in the form of ansatz A_4. Thus, the corresponding model function $F_4(X,Y,Z)$ is constituted by 35 monomials. This model function is not reduced very much and no differential model has been obtained from the w variable of the Rössler system. Obtaining a three-dimensional differential model from this variable is a great challenge. So far, no three-dimensional differential model has been obtained using a general polynomial expansion [3]. It has been shown that such a feature results from a lack of observability of the dynamics when the w variable is used [21]. Nevertheless, when a rational function is used with the right denominator selected with a fixed point identification, a three-dimensional differential model can be obtained [8]. In our case, it seems that the ansatz A_4 is still too general to allow a successful differential model. Further investigations are postponed for future works.

When v is the observable, the Rössler system may be directly compared to the ansatz A_2 for which all coefficients are zero except those involved in ansatz A_0 which is a subpart of ansatz A_2. A_2 is therefore the right ansatz for describing the Rössler system. In order to check whether our method is able to select it, we try to obtain differential models using ansatz A_1 and A_2. Although differential models can be obtained by using both ansatz (Fig. 3), they are not of the same quality. Indeed, the obtained model with the wrong ansatz [ansatz A_1, Fig. 4(a)] generates a chaotic attractor which is less developed than the one obtained with the adequate ansatz [ansatz A_2, Fig. 4(b)]. This is observed more clearly when a first-return map to the Poincaré section

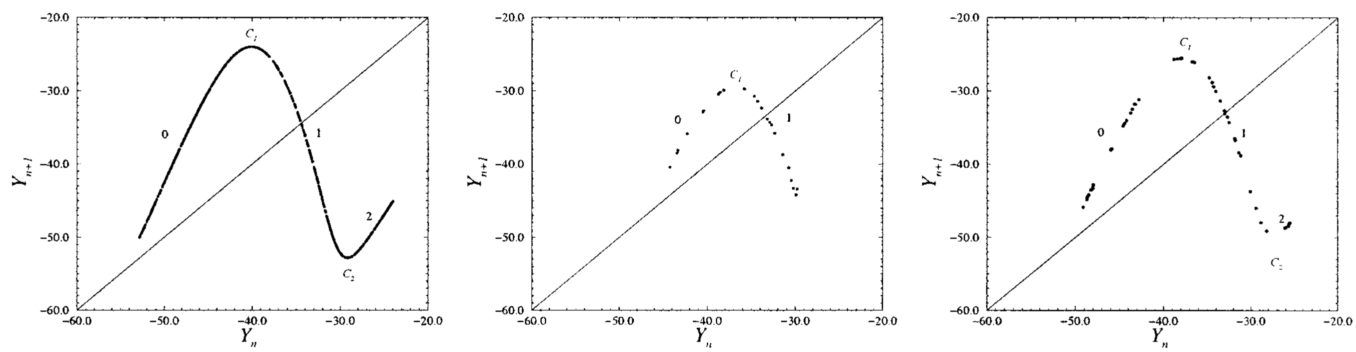
$$P \equiv \{(Y_n, Z_n) \in \mathbb{R}^2 | X_n = 0, \dot{X}_n < 0\} \tag{34}$$

is computed. The three branches observed on the first-return map of the exact differential model [Fig. 4(a)] are only reproduced by the differential model associated with ansatz A_2. Moreover, the differential model associated with ansatz A_1 is numerically unstable and only a metastable chaotic behavior is observed before the trajectory escapes to infinity. The differential model associated with ansatz A_2 generates a limit cycle after a transient regime. At first sight, one may conclude that the dynamical behavior associated with the ansatz model may be very different from the one generated by the original Rössler system. But the limit cycle corresponds to a periodic window which is very close to the original chaotic behavior in the bifurcation diagram of the Rössler system. Such a departure may result from a slight change in the control parameter values and may be quantified by inspecting the symbolic sequence of the last created periodic orbit which is $(\overline{202001})$ for the original Rössler system and $(\overline{20201})$ for the ansatz model. When b and $\mathcal{C}$ are kept unchanged, it corresponds to an a value equal to 0.197 rather than 0.200. Such a slight departure for the control parameters (less than 1.5%) is a natural consequence of the application of an estimation algorithm to data records that include numerical errors.

A definite selection of ansatz A_2 is done by trying to invert numerically the map φ. Only the map φ_2 associated with ansatz A_2 can be inverted. Indeed, the map φ_1 is not invertible because the leading coefficients, a_2 and b_6, involved in ansatz A_1 are equal to zero and, consequently, imply divisions by zero. As expected, only ansatz A_2 is adequate for modeling the v-induced Rössler attractor. The ansatz model reads

$$\begin{aligned}
\dot{x} &= 26.4534 + 0.0519x + 8.5377y, \\
\dot{y} &= 0.4573 - 0.1163x + 0.1476y - 0.0001z, \\
\dot{z} &= -0.4414 - 8.8897x - 18.0548y - 8.5462z \\
&\quad - 0.1392x^2 + 2.7345xy - 0.1481xz + 7.0057y^2 \\
&\quad + 8.5377yz + 0.0000z^2.
\end{aligned} \tag{35}$$

Extra terms are involved in this ansatz model. They result from the extra terms involved in ansatz A_2. Nevertheless, the chaotic attractor [Figs. 1(g), 1(h), and 1(i)] generated by the



(a) Exact differential model (b) A_1 differential model (c) A_2 differential model

FIG. 4. First-return map for the exact differential model derived from the Rössler system and for the two differential models obtained with the two different ansatz. Trajectory may be encoded according to the generating partition defined by the two critical points C_1 and C_2. Symbols 0, 1, and 2 are associated with the increasing, as well as the decreasing and increasing, branches, respectively. The bad quality of the A_1 differential model is characterized by the absence of the third monotonic branch on the first-return map.

ansatz model presents the same orientation in the space (x,y,z) as the original Rössler attractor in the original phase space spanned by (u,v,w). This can be checked by comparing Figs. 1(a), 1(b), and 1(c) with Figs. 1(g), 1(h), and 1(i). This means that the couplings between dynamical variables (x,y,z) are very similar to the couplings between the original dynamical variables (u,v,w).

The quality of the ansatz model is confirmed by its first-return map which compares favorably to the first-return map computed for the original Rössler system [Fig. 4(a)]. The original first-return map is equivalent to the one computed for the differential model. As observed for the differential model, the first-return map associated with the ansatz model is not very well visited. In fact, the ansatz model generates a limit cycle after a transient regime as observed for the differential model. Therefore, the dynamics is preserved under the transformation back to the phase space $\mathbb{R}^3(x,y,z)$ and the model now has the great advantage of presenting couplings between the dynamical variables which are similar to those between the original dynamical variables (u,v,w).

The ansatz model captures the right properties of the dynamics. Although the x variable of the ansatz model evolves within the same range as the v variable of the original Rössler system, the other variables (y and z) of the ansatz model are rescaled due to the departure observed for the estimated control parameter b. (See Fig. 5.)

B. Lorenz system

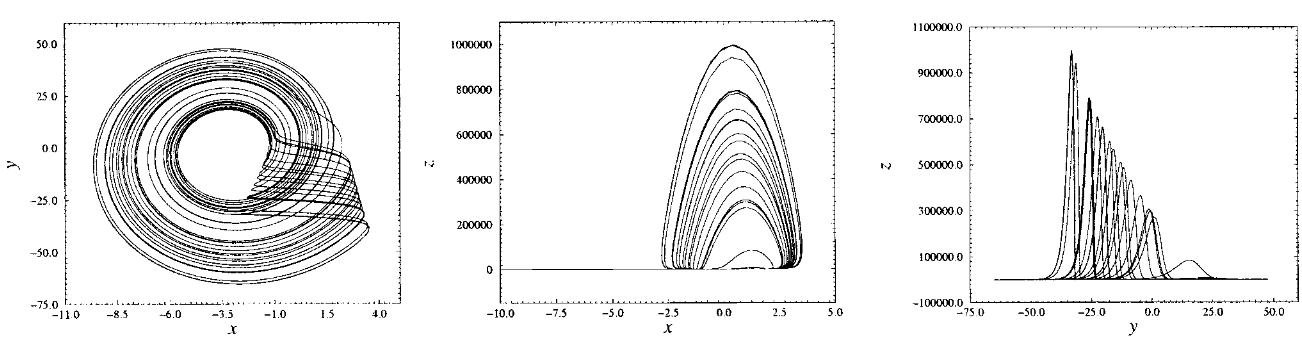
The second test case is the Lorenz system [13] reading as

$$\dot{u}=-\sigma u+\sigma v,$$

$$\dot{v}=\mathcal{R}u-v-uw, \tag{36}$$

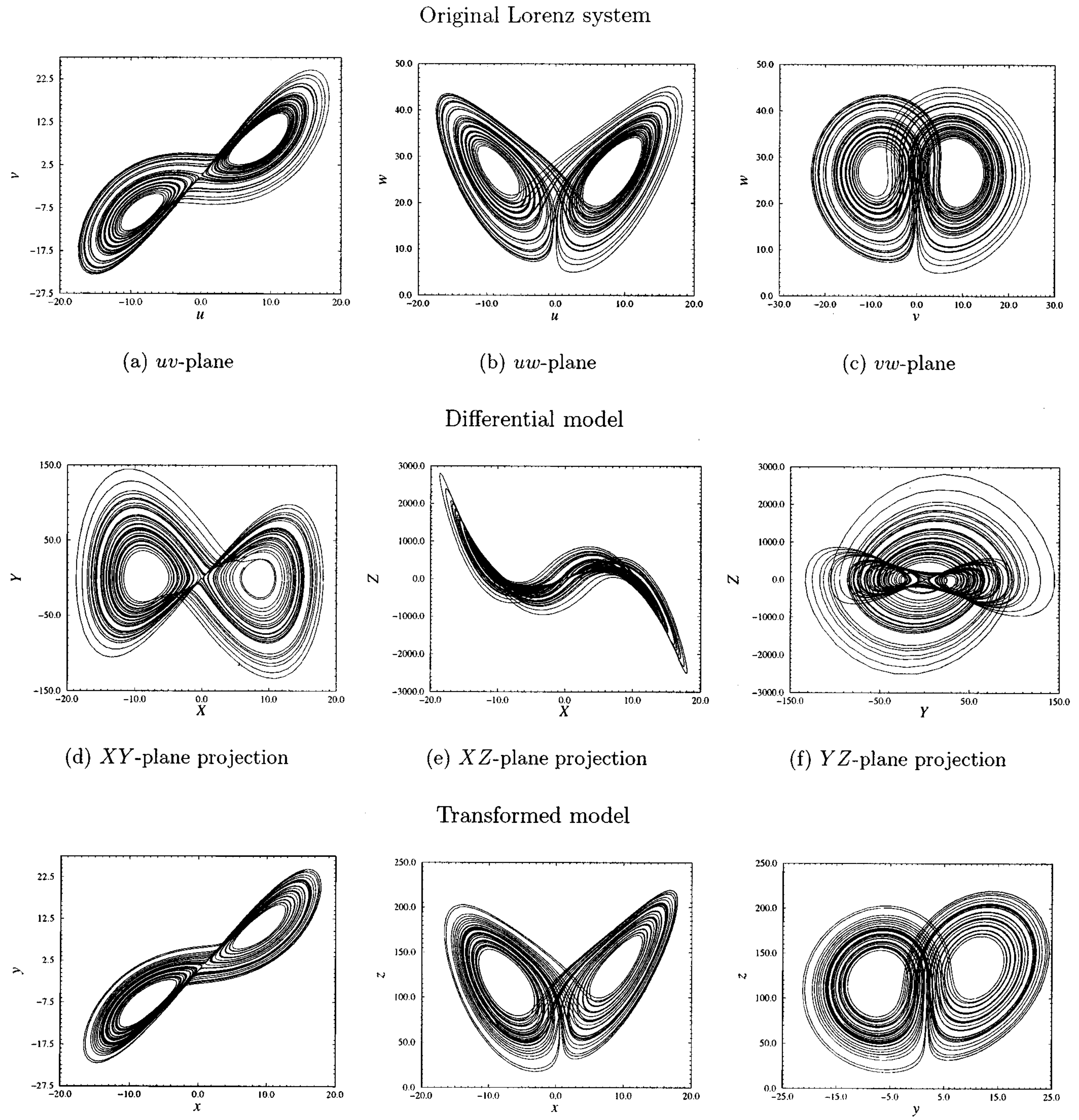
$$\dot{w}=-bw+uv,$$

where $(\mathcal{R},\sigma,b)=(28.0,10.0,\frac{8}{3})$ are the control parameters. Plane projections of the attractor generated by this system are displayed in Figs. 6(a), 6(b), and 6(c). Since we know the original equations, we can check that when u is the observable, the Lorenz system corresponds to ansatz A_1 with



(a) yx-plane (b) xz-plane (c) yz-plane

FIG. 5. Plane projections of the attractor generated by the ansatz model obtained from the v variable of the Rössler system by using ansatz A_2. The scales of the variables (x,y,z) are different from those for the Rössler system due to the departure for the c_0 value.



(a) uv-plane (b) uw-plane (c) vw-plane

(d) XY-plane projection (e) XZ-plane projection (f) YZ-plane projection

(g) xy-plane projection (h) xz-plane projection (i) yz-plane projection

FIG. 6. Plane projections of the attractor generated by the original Lorenz system, the differential model obtained using ansatz A_1, and the ansatz model. The right symmetry properties are recovered for the ansatz model.

$$\begin{aligned} &a_1=-\sigma, \quad a_2=\sigma, \\ &b_1=R, \quad b_2=-1, \quad b_6=1, \\ &c_3=-b, \quad c_5=1. \end{aligned} \tag{37}$$

but not to ansatz A_2. In the former case, we have $(x,y,z)=(u,v,w)$.

Since the exact form of the Lorenz system is known, the exact model function F_L reads

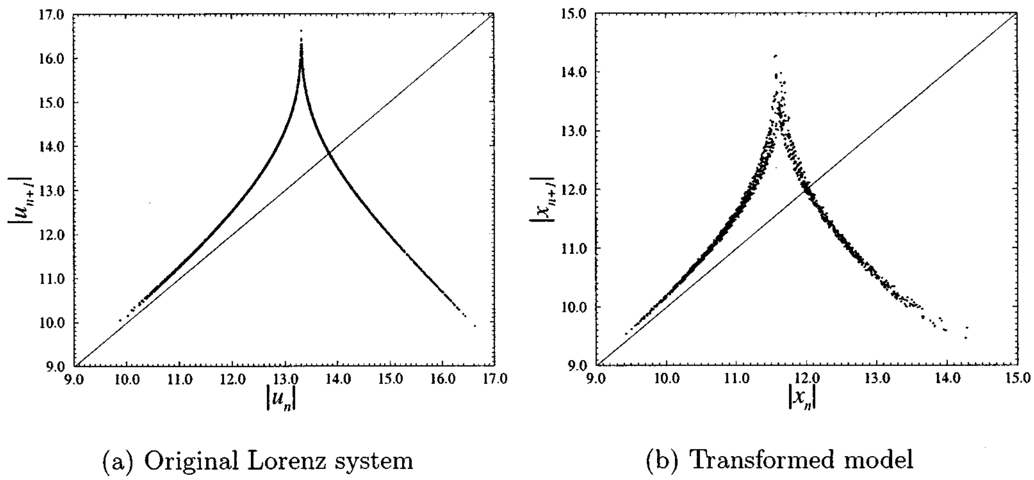


FIG. 7. First-return maps for the Lorenz case. The map associated with the ansatz Lorenz model exhibits slight departures from exact symmetry which induce a significant thickness.

$$F_L = \alpha_3 X + \alpha_5 X^3 + \alpha_{10} Y + \alpha_{13} X^2 Y + \alpha_{18}\frac{Y^2}{X} + \alpha_{26} Z + \alpha_{32}\frac{YZ}{X}, \tag{38}$$

according to ansatz A_1. One may remark that this exact model function is constituted by seven terms rather than 35 for the model function F_1 associated with ansatz A_1. There is a cost to pay for these additional terms since they will slightly affect the quality of the model as will be discussed later.

Two differential models of the form (3) are obtained with ansatz A_1 and A_2, respectively. For instance, plane projections of the chaotic attractor generated by the differential model with ansatz A_1 are displayed in Figs. 6(d), 6(e), and 6(f). The model function is constituted here by 35 terms rather than 18 as obtained in [3]. The great advantage of the model function is that it has a structure which is equivalent to those of the exact model function F_L, while the estimated model function by using a polynomial expansion has not. One may observe the large departure from the original attractor displayed in Fig. 6. Indeed, the couplings between the derivative coordinates are clearly different from the couplings between the original dynamical variables (u,v,w). The symmetry of the differential model is an inversion symmetry with respect to the origin of the differential embedding rather than a rotation [14].

Only the differential model associated with the ansatz A_1 can be transformed into an ansatz model of the form

$$\begin{aligned}
\dot{x} &= -1.799 - 22.605x + 21.650y + 0.001x^2,\\
\dot{y} &= -0.961 - 0.300x + 11.617y - 0.100x^2 \\
&\quad + 0.104xy - 0.160xz + 0.026y^2,\\
\dot{z} &= -13.811 + 15.518x - 3.494y - 2.552z + 0.373x^2 \\
&\quad - 1.445xy - 0.153xz + 3.748y^2 - 0.006yz \\
&\quad - 0.001z^2.
\end{aligned} \tag{39}$$

Its numerical integration generates a chaotic attractor [Figs. 6(g), 6(h), and 6(i)] which may be favorably compared to the original Lorenz attractor displayed in Figs. 6(a), 6(b), and 6(c). The x or y variable generates time series with symmetry properties while the z variable is clearly invariant. This is an important result since by using derivatives, delay coordinates, or principal components, the reconstructed attractor always possesses an inversion symmetry when u or v are used as the observable while no symmetry is observed when w is used as the observable [14]. The ansatz model, therefore, has the great advantage of generating a phase portrait that has a rotation symmetry as observed for the original phase portrait. Using an ansatz allows one to select the adequate model structure and to recover the exact symmetry properties. One may remark that a rotation symmetry, has been recovered when starting from a single time series while two time series are required by using usual approaches [18,19].

The ansatz model (39) is not exactly equivariant. Indeed, extra terms required by the ansatz A_1 as a_0, a_7x^2 or b_4xy and other even order monomials involving x and y slightly destroy the equivariance since $\dot{x}$ and $\dot{y}$ can no longer be changed into $-\dot{x}$ and $-\dot{y}$ by applying $(x,y,z)\rightarrow(-x,-y,z)$. These extra terms have low effects when the system is integrated as displayed in Figs. 6(g), 6(h), and 6(i). The thickness of a first-return map exhibits this slight departure from symmetry (Fig. 7). It could be avoided with a restricted ansatz only including odd terms in the first two equations, i.e., by deleting terms associated with a_4, b_4, b_5, and b_7 from the ansatz before estimating the differential model.

When the v or w variable is the observable, no ansatz models can be obtained with the ansatz library introduced in Sec. II.

IV. CONCLUSION

Modeling dynamical systems starting from a scalar time series is an important subject of research. Most of the models obtained involve polynomial expansions with a large number of coefficients and the structure of the model equations cannot be used to understand the couplings between the different dynamical variables required for a complete description of the state of the system studied. With the aid of an ansatz library, it is possible to reduce the number of terms involved in the model function when its structure is adequate. Moreover, when the original system presents a symmetry, the differential model cannot be characterized by the same symmetry except if the symmetry of the original system is an inversion. We have shown that using an ansatz library may allow us to obtain an ansatz model that has variables which present couplings very similar to those observed for the original system. For instance, the right symmetry properties of the Lorenz system have been recovered starting from a scalar time series. The first attempts to adopt this method for noisy time series has been postponed for future research.

ACKNOWLEDGMENTS

We wish to thank Luis A. Aguirre for helpful discussions and Irina Gorodnitsky for encouraging this project. This project was supported by an AMADEUS program between Austria and France and by NSF Grant No. IIS-0082119. C.L. was at the Institut fur Theoretische Physik in Graz, Austria, when most of this work was done.

APPENDIX: COMPUTATION OF MAP φ_p FOR ANSATZ A_p

In this appendix the code used in MATHEMATICA for computing the map φ between the ansatz coefficient α_n's and the model coefficients $\{a_i, b_j, c_k\}$ is given for ansatz A_0. Other cases can be easily computed from this code.

General procedure to generate the functions f_i

Matrices AA, BB, and CC correspond to the ansatz functions f_1, f_2, and f_3, respectively. The matrix indices are shifted by 1. For instance, in matrix *AA*, the indices 2 and 3 correspond to coefficients a_1 and a_2, respectively,

$$\mathtt{Q}=\{1,x[t],y[t],z[t],x[t]x[t],x[t]y[t],x[t]z[t],y[t]y[t],y[t]z[t],z[t]z[t]\},$$

$$\mathtt{AA}=\text{Flatten}[\text{Table}[\{a_i\},\{i,0,\text{Length}[\mathtt{Q}]-1\}]],$$

$$\mathtt{BB}=\text{Flatten}[\text{Table}[\{b_i\},\{i,0,\text{Length}[\mathtt{Q}]-1\}]],$$

$$\mathtt{BB}=\text{Flatten}[\text{Table}[\{b_i\},\{i,0,\text{Length}[\mathtt{Q}]-1\}]],$$

$$\mathtt{CC}=\text{Flatten}[\text{Table}[\{c_i\},\{i,0,\text{Length}[\mathtt{Q}]-1\}]],$$

$$\mathtt{T}=\mathtt{AA}\;\mathtt{Q;T}=\mathtt{T}[[\{2,3\}]];f_1=\text{Apply}[\text{Plus},\mathtt{T}],$$

$$\mathtt{T}=\mathtt{BB}\;\mathtt{Q;T}=\mathtt{T}[[\{2,4\}]];f_2=\text{Apply}[\text{Plus},\mathtt{T}],$$

$$\mathtt{T}=\mathtt{CC}\;\mathtt{Q;T}=\mathtt{T}[[\{1,4,9\}]];f_3=\text{Apply}[\text{Plus},\mathtt{T}],$$

$$a_1\,x[t]+a_2\,y[t],$$

$$b_1\,x[t]+b_3\,z[t],$$

$$c_0+c_3\,z[t]+c_8\,y[t]z[t].$$

Computation of $\dot{Z}$

$$\mathtt{replxyz}=\{x[t]\to x,y[t]\to y,z[t]\to z\},$$

$$g_1=x[t],$$

$$g_2=\text{Simplify}[(\partial_{x[t]}g_1)f_1+(\partial_{y[t]}g_1)f_2+(\partial_{z[t]}g_1)f_3],$$

$$g_3=\text{Simplify}[(\partial_{x[t]}g_2)f_1+(\partial_{y[t]}g_2)f_2+(\partial_{z[t]}g_2)f_3],$$

$$\text{Zdot}=\text{Simplify}[(\partial_{x[t]}g_3)f_1+(\partial_{y[t]}g_3)f_2+(\partial_{z[t]}g_3)f_3],$$

$$g_1=g_1\,/.\text{replxyz},\quad g_2=g_2\,/.\mathtt{replxyz},\quad g_3=g_3\,/.\mathtt{replxyz};$$

$$\text{Zdot}=\text{Zdot}/.\mathtt{replxyz},$$

$$\mathtt{S}=\text{Flatten}[\text{Solve}[\{g_1==X,g_2==Y,g_3==Z\},\{x,y,z\}]],$$

Zdot=ExpandAll[Zdot/S].

Extraction of the monomial list from the expression for $\dot{Z}$

repl=Flatten[Table[{$a_i\rightarrow 1$,$b_i\rightarrow 1$,$c_i\rightarrow 1$},{i,0,Length[Q]}]];

QQ=ExpandAll[Simplify[(Zdot/.repl))]];

QQQ=Table[0,{i,1,Length[QQ]}];L=QQ:

For[$j=1$,$j\leqslant$Length[QQ],$j++$,

If[NumberQ[QQ[[j]]]==False,

For[$k=1$,$k\leqslant$Length[QQ[[j]]],$k++$,

If[NumberQ[QQ[[j,k]]]==True,QQQ[[j]]+=QQ[[j,k]]];

];

QQQ[[j]]+=QQ[[j]];

];

L[[j]]/=QQQ[[j]];

L=Table[L[[i]],{i,1,Length[L]}],

$\{1, X, X^2, Y, X, Y, Y^2, Z, X, Z, Y\, Z\}$.

Extraction of the coefficients α_n from $\dot{Z}$ using the monomial list

Coeff=Table[0,{i,1,Length[L]}];

pp=Length[L];

For[j=Length[L],$j>0$,$j--$,

If[NumberQ[L[j]]/.repl]==False,

For[$k=1$,$k\leqslant$Length[LZ],$k++$,

CO=Coefficient[LZ[[k]],L[[j]]];

If[NumberQ[CO/.repl]==True,Coeff[[j]]+=CO;]

];

$jj=j$;

];

];

$$\text{Coeff}[[jj]]=\text{ExpandAll}[\text{Simplify}[\text{Apply}[\text{Plus},\mathsf{LZ}]-\text{Apply}[\text{Plus},L\ \text{Coeff}]]].$$

Listing of the transformation φ

$$\text{For}[k=1,k\leqslant\text{Length}[L],k++,\text{Print}[\alpha\_,k,=\text{Coeff}[[k]]];$$

$$\alpha\_1=a_2b_3c_0,$$

$$\alpha\_2=-a_2b_1c_3,$$

$$\alpha\_2=-a_2b_1c_3,$$

$$\alpha\_3=a_1b_1c_8,$$

$$\alpha\_4=a_2b_1-a_1c_3,$$

$$\alpha\_5=\frac{a_1^2c_8}{a_2}-b_1c_8,$$

$$\alpha\_6=-\frac{a_1c_8}{a_2},$$

$$\alpha\_7=a_1+c_3,$$

$$\alpha\_8=-\frac{a_1c_8}{a_2},$$

$$\alpha\_9=\frac{c_8}{a_2}.$$

[1] F. Takens, in *Dynamical Systems and Turbulence, Warwick, 1980*, edited by D. A. Rand and L. S. Young, Lecture Notes in Mathematics Vol. 898 (Springer-Verlag, New York, 1981), pp. 366–381.

[2] R. Brown, N. F. Rul'kov, and E. R. Tracy, Phys. Rev. E **49**, 3784 (1994).

[3] G. Gouesbet and C. Letellier, Phys. Rev. E **49**, 4955 (1994).

[4] K. Judd and A. I. Mees, Physica D **82**, 426 (1995).

[5] J. B. Kadtke and J. S. Brush, (unpublished).

[6] O. Anosov, O. Butkovskii, and Y. Kravtsov, *Predictability of Complex Dynamical Systems* (Springer-Verlag, New York, 1996), p. 105.

[7] L. A. Aguirre and S. A. Billings, Int. J. Control **62**, 569 (1995).

[8] L. Le Sceller, C. Letellier, and G. Gouesbet, Phys. Rev. E **60**, 1600 (1999).

[9] G. Gouesbet and J. Maquet, Physica D **58**, 202 (1992).

[10] S. A. Billings, S. Chen, and J. M. Korenberg, Int. J. Control **49**, 2157 (1989).

[11] L. A. Aguirre, U.S. Freitas, C. Letellier, L. Le Sceller, and J. Maquet, in *Stochaos: Stochastic and Chaotic Dynamics in the Lakes*, edited by D. S. Broomhead, E. A. Luchinskaya, P. V. E. McClintock, and T. Mullin (AIP, Woodbury, NY, 2000), pp. 649–655.

[12] L. A. Aguirre, U.S. Freitas, C. Letellier, and J. Maquet (unpublished).

[13] E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).

[14] C. Letellier and G. Gouesbet, J. Phys. II **6**, 1615 (1996).

[15] C. S. M. Lainscsek, F. Schürrer, and J. B. Kadtke, Int. J. Bifurcation Chaos Appl. Sci. Eng. **8**, 899 (1998).

[16] N. H. Packard, J. P. Crutchfield, J. D. Farmer, and R. S. Shaw, Phys. Rev. Lett. **45**, 712 (1980).

[17] O. E. Rössler, Phys. Lett. A **57**, 397 (1976).

[18] G. P. King and I. Stewart, Physica D **58**, 216 (1992).

[19] C. S. M. Lainscsek, F. Schürrer, and J. B. Kadtke, *Proceedings of the International Conference on Unsolved Problems of Noise, Szeged, Hungary, 1996* (World Scientific, Singapore, 1997).

[20] R. Thomas, J. Biol Syst **7**, 225 (1999).

[21] C. Letellier, J. Maquet, L. Le Sceller, G. Gouesbet, and L. A. Aguirre, J. Phys. A **31**, 7913 (1998).