Erzwungene Schwingungen bei veränderlicher Eingenfrequenz und ihre technische Bedeutung

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Excerpt translated by Otto E. Rössler

[...]

23. The Equations of Motion of the Experimental System can easily be written down using Figure 14.

In the equilibrium position of the small pendulum (under the angle Ψ_0 relative to the vertical) we have

$$0 = g\,S\,\sin\Psi_0 + c(T_{20} - T_{10} - T_{30})\,.$$

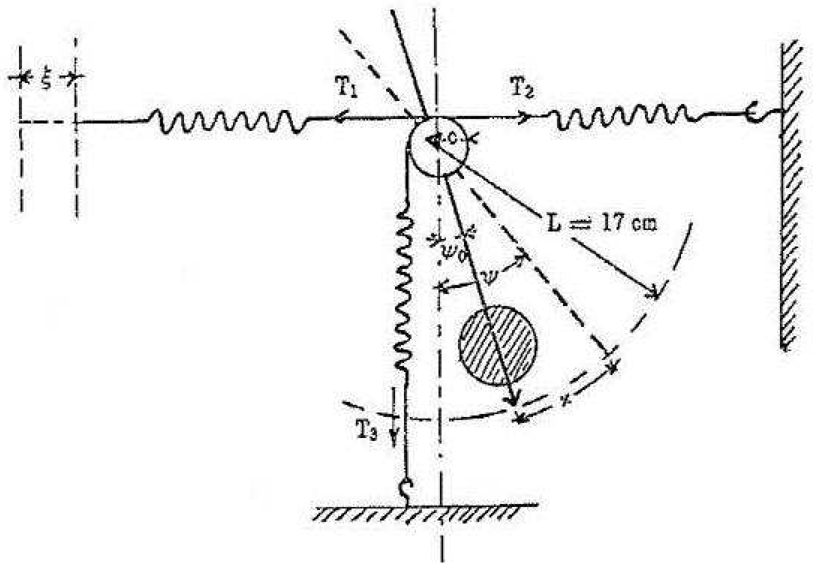

Fig. 14.

Here gS is the static momentum of the pendulums body (with drum and weights), c is the radius of the driving drum, T_{10}, T_{20}, T_{30} are the tensions of the fibers. If one denotes by J the moment of inertia of the mass of the pendulum relative to the axis of rotation, then in the case of motion under the angle Ψ, the "delaying momentum" [Verzgerungsmoment] is

$$-J\frac{\mathrm{d}^2\Psi}{\mathrm{d}t^2} = gS\sin\Psi + c(T_2 - T_1 - T_3)\,.$$

If f_1, f_2, f_3 are the constants of the three springs, one has

$$\begin{aligned} T_1 - T_{10} &= f_1[\xi - c(\Psi - \Psi_0)]\,, \\ T_2 - T_{20} &= f_2 c(\Psi - \Psi_0)\,, \\ T_3 - T_0 &= -f_3 c(\Psi - \Psi_0)\,. \end{aligned}$$

From this, one obtains

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}t^2} + \frac{gS}{J}(\sin\Psi - \sin\Psi_0) + \frac{c^2}{J}(f_1 + f_2 + f_3)(\Psi - \Psi_0) = \frac{cf_1}{J} \cdot \xi$$

as the equations of motion of the pendulum, if in addition one takes into regard the law under which the forcing motion ξ takes place.

Given the relatively large mass of the pendulum B, one can neglect the back-action exerted by S on B, and is allowed to assume that ξ is the changing in time, independently of the motion of the small experimental pendulum.

The law governing the forced motion with a very good approximation has the form $\xi = a \sin\omega t$, where a is a constant that is given by the respective maximal excursion of pendulum B.

Introducing

$$x = L(\Psi\Psi_0)$$

and

$$\frac{x}{L}\cos\Psi_0 - \frac{x^2}{2L^2}\sin\Psi_0 - \frac{x^3}{6L^3}\cos\Psi_0$$

in place of $\sin\Psi\sin\Psi_0$, one obtains the law of motion in the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \left[\frac{gS}{L}\cos\Psi_0 + \frac{c^2}{J}(f_1 + f_2 + f_3)\right]x - \frac{1}{2L}\frac{gS}{J}\sin\Psi_0 \cdot x^2 - \frac{1}{6L^2}\frac{gS}{J}\cos\Psi_0 \cdot x^3 = \frac{cf_1L}{J}a\sin\omega t$$

or

$$\ddot{x} + \alpha x - \beta x^2 - \gamma x^3 = k\sin\omega t\,,$$

whereby

$$\begin{cases} \alpha = \dfrac{gS}{J}\cos\Psi_0 + \dfrac{c^2}{K}(f_1 + f_2 + f_3)\,, \\ \beta = \frac{1}{2L}\frac{gS}{J}\sin\Psi_0\,, \\ \gamma = \frac{1}{6L^2}\frac{gS}{J}\cos\Psi_0\,, \\ k = \frac{cf_1L}{J}a\,. \end{cases}$$

The coefficient of α is easy to determine experimentally by ones letting the system perform very small proper vibrations and obtaining the duration of one period.

If one removes all threads from the drum, then one gets $\Psi_0 = 0$, and from the derived α one obtains the value $\frac{gS}{J}$. When one then restores the connections with the threads, to observe *small* eigen vibrations ($k = 0$), one obtains the value $\frac{c^2}{J}(f_1 + f_2 + f_3)$. The values β, γ, are easy to obtain by calculation.

The value $x = s$ is to be observed on the reading scale of the small pendulum to calculate k. In most cases it suffices to assume $k = \alpha s$. Then the value $\frac{k}{\alpha}$

from Eq. (69) is nothing else but the static excursion on the reading scale that corresponds to the maximal value of ξ.

To measure the tensions of the springs, the weights and the moments of inertia proved unnecessary. All that needed to be determined was the vibration numbers of the experimental pendulum valid at very small excursions, and this — first inside the system when it is isolated; then the vibration numbers of the forcing pendulum at the excursions used; then the static excursions of the experimental pendulum corresponding to the respective maximal ξ, and the dynamical excursions in the state of oscillation. The excursions of the experimental pendulum are indicated in the state of oscillation. The excursions of the experimental pendulum are indicated in millimeters, measured along a circular scale of radius 170 mm.

From the series of experiments done, here only two are to be highlighted.

[...]

IV. Influence of the Damping

26. The experiments showed that even a small damping has a visible influence on the magnitude of the excursion, whereas in a quantitative respect, a weak damping does not make itself felt. The task is now to make this state of affairs amenable to quantitative calculation.

That in this case at least the calculational amount of work would be markedly increased, was to be expected *a priori*. Therefore, here only the case of the forced symmetric pseudoharmonic vibration is to be treated; its equation of motion is

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \chi\frac{\mathrm{d}x}{\mathrm{d}t} + \alpha x - \gamma x^3 = k\sin\omega t\,. \tag{1}$$

We here again at first look in passing at the equation of the damped harmonic vibration,

$$\ddot{x} + \chi\dot{x} + \alpha x = k\sin\omega t \tag{2}$$

and try to solve it in the same way as in the case of the undamped vibration, by means of successive approximation.

[...]

An experimental confirmation of the results of the calculation presented in Fig. 17 would be extraordinarily desirable. It can be obtained by means of larger pendulum-devices which enable a recording of vibration diagrams; or through measurements made on electromagnetic vibration circuits. In the case of the latter method, special attention would need to be placed on the excitation corresponding with maximum possible exactness to the form $k \sin \omega t$, so that the experimental results can be compared directly with our calculated results. The conditions of [ou]r time unfortunately did not allow me to follow up on the one or the other way further on.

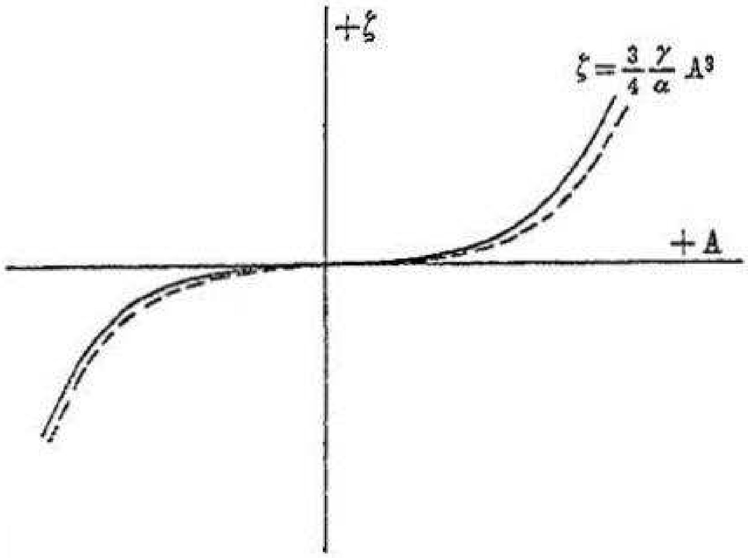

Fig. 17.