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Characterization of the Lorenz system, taking into account the equivariance of the vector field

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We characterize the chaotic attractors of the Lorenz system associated with $R=28$ and $R=90$ (reduced Rayleigh number) by using a partition that takes into account the equivariance of the vector field. The population of unstable periodic orbits is extracted and encoded respectively with binary and three letter symbolic dynamics. Templates are proposed for these R values.

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I. INTRODUCTION

In the past few years several works discussed the topological description of chaotic attractors. In particular, the idea has arisen that an attractor can be described by the population of periodic orbits, their related symbolic dynamics, and their linking numbers [1]. In three dimensional cases, periodic orbits may be viewed as knots [2] and, consequently, they are robust with respect to smooth parameter changes and allow the definition of topological invariants under isotopy.

A topological analysis procedure may consist of a number of steps. A population of unstable periodic orbits is first extracted from the flow. Then, the topological organization of the unstable periodic orbits is determined by computing interlinking and self-linking numbers. From such an analysis of a few small period orbits, a template is built. This template can be used to predict the linking numbers characterizing the orbits. The comparison between template predictions and topological invariant measurements provides a checking of the template prediction.

Although the case of asymmetric systems is well documented, such a topological characterization is not fully understood for equivariant systems. For instance, the Lorenz system template proposed by Mindlin *et al.* [1] is not consistent with the Lorenz map [3]. However, the Mindlin *et al.* template corresponds to another set of control parameters, after the homoclinic explosion. Indeed, this template is composed of two bands without any local torsion, thus conflicting with the existence of the decreasing monotonic branch of the Lorenz map (a decreasing branch in a map must be associated with a band whose local torsion is odd [2]). To solve this contradiction, we propose an equivariant description of the Lorenz attractor. A binary symbolic dynamics, based on the Lorenz map, is used to encode all orbits extracted from the attractor up to period 8 (for the reduced Rayleigh number $R=28$). An equivariant template is then extracted and checked. Also, a three letter equivariant dynamics is proposed for more developed chaos ($R=90$), allowing one to encode the population of unstable orbits. The corresponding template is again extracted and checked.

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II. EQUIVARIANT CHARACTERIZATION OF THE LORENZ SYSTEM

A. Vector field equivariance

A vector field $f(\boldsymbol{\lambda},\mathbf{x}(t))$ is equivariant if

$$\mathbf{f}(\boldsymbol{\lambda},\gamma\mathbf{x}(t))=\gamma\,\mathbf{f}(\boldsymbol{\lambda},\mathbf{x}(t)) , \tag{1}$$

in which $\mathbf{x}(t)$ is a real-valued vector, t is the time, $\boldsymbol{\lambda}$ is a parameter vector, $\mathbf{f}$ is a smooth vector-valued function, and γ is a matrix defining the equivariance.

The Lorenz system [3] with parameter vector $\boldsymbol{\lambda}=(R,\sigma,b)$ and variables $\mathbf{x}=(x,y,z)$, and the usual notations, is equivariant with the equivariant matrix reading

$$\gamma=\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{2}$$

The Lorenz equivariant is a Z_2 symmetry, i.e., $\gamma^2=\underline{I}$. The Lorenz system remains unchanged if $\mathbf{x}$ is replaced by $\gamma\mathbf{x}$, i.e., if $\mathbf{x}(t)$ is a solution, then $\gamma\mathbf{x}(t)$ is also a solution. Periodic orbits of the Lorenz system may be symmetric or asymmetric (degenerate in Cvitanovič terminology [4]) with respect to the Z_2 symmetry. Symmetric orbits are globally invariant under the action of γ. Asymmetric orbits are mapped to their symmetric configuration and therefore appear by pairs.

B. First-return maps

The strange chaotic Lorenz attractor for $\boldsymbol{\lambda}=(28,10,\frac{8}{3})$ is organized around three fixed points $C_0(x=y=z=0)$ and

$$C_\pm(x_\pm=\pm\sqrt{b(R-1)},y_\pm=x_\pm,z_\pm=R-1) .$$

The Lorenz map reads $M_{n+1}=g(M_n)$, in which M_n is the nth z maximum of the time series. These maximum values may be obtained from the intersection of the trajectory with two hypersurfaces Σ_+ and Σ_- defined by

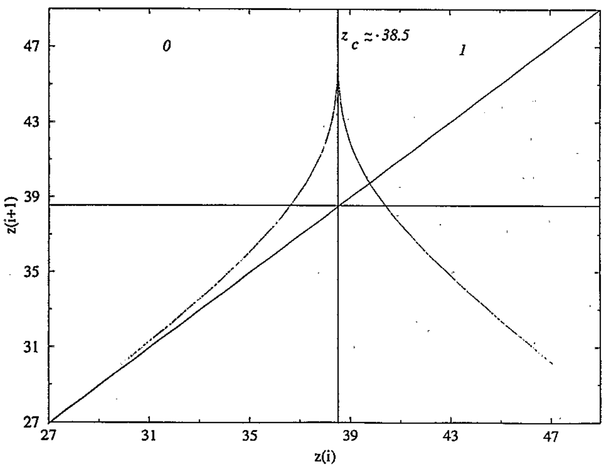

FIG. 1. First-return map from P_y to itself: $z(i+1)$ versus $z(i)$ ($R=28$).

$$\Sigma_{\pm}=\left\{(x,y,z)\in R^3 \,\middle|\, \frac{\partial f}{\partial z}=0, \frac{\partial^2 f}{\partial z^2}<0, x \gtrless 0, y \gtrless 0\right\}. \tag{3}$$

Each hypersurface is a Poincaré section. When constructing the Lorenz map, these sections are not distinguished as a consequence of the Z_2 symmetry. Therefore, the map is constructed by using a Poincaré set $\Sigma=\Sigma_+ \cup \Sigma_-$ rather than a Poincaré section. Rather than using Σ_+ and Σ_-, it is numerically more convenient to use the Poincaré sections P_{y+} and P_{y-} defined by

$$P_{y\pm}=\left\{(x,y,z)\in R^3 | y=y_{\pm}, \frac{\partial f}{\partial y} \gtrless 0\right\}. \tag{4}$$

The first-return map to the Poincaré set $P_y=P_{y+}\cup P_{y-}$, equivalent to the Lorenz map, is displayed in Fig. 1 using the coordinate z.

C. Population of unstable periodic orbits

Let us consider a two-dimensional (2D) map with coordinates (α,β), initial values of an orbit (α_0,β_0), and the nth iterate (α_n,β_n). The Euclidean distance $d_n(\alpha_0,\beta_0,\alpha_n,\beta_n)$ defines a surface S associating an altitude d_n with each point (α_0,β_0). Starting from $d_n>0$, a research algorithm allows us to reach the sea floor $d_n=0$, therefore locating an n-period orbit. This descent method has been successfully checked in the case of the Hénon map by comparison against the results of Biham and Wenzel [5]. For the Lorenz flow here studied, n-period orbits are located by using the 2D map generated from the Poincaré set P_y.

The population is extracted for $(R,\sigma,b)=(28,10,\frac{8}{3})$. With the encoding partition given by letter 0 if $z<z_c$ and letter 1 if $z>z_c$, the population of periodic orbits is displayed in Table I up to $n=4$ ($n=8$ periods are also available), the coordinates (x,z) corresponding to the rightmost periodic point in Fig. 1. Symmetric orbits go back to their initial conditions after following the symbolic sequence (WW), i.e., twice the encoding symbolic sequence (W) for this orbit [4]. Therefore, their encoding sequences display an odd number of 1's.

D. Template

To extract the template, we start from the two symmetric wing Lorenz mask [Fig. 2(a)] used by Birman and Williams [7]. Each wing contains two bands, one labeled 0, associated with the increasing monotonic branch of the Lorenz map and the other one, labeled 1, associated with the decreasing branch of the Lorenz map, leading to the four band template given in Fig. 2(b).

Following Cvitanovic and Eckard [4], such a four band template may be reduced to a two band template corresponding to an equivariant fundamental domain, i.e., a single wing. Choosing the left wing in Fig. 2(b), we then obtain the template in Fig. 2(c). The reinsertion of band 1 in Fig. 2(b) produced a rotation of ($+\pi$) for band 1 (i.e., a local torsion of $+1$) in Fig. 2(c) (a positive local torsion is associated with a clockwise rotation [2]) introduced by the action of matrix γ. The standard insertion convention is used [7]. The obtained template is then in agreement with the Lorenz map: a band 0 (even local torsion) is associated with the increasing (orientation preserving) branch of the map and a band 1 (odd local torsion) is associated with the decreasing (orientation reversing) branch of the map.

A template may be described by a template matrix M_{ij} in which M_{ij} is the local torsion of the ith band if $i=j$, and the sum of the oriented crossings between the ith and jth bands if $i\neq j$. We then have

$$M_{ij}=\begin{bmatrix}0 & 0\\ 0 & +1\end{bmatrix}. \tag{5}$$

E. Template checking

Template checking can be carried out by considering one pair of orbits, namely, (101,10) of the form (N_1,N_2).

TABLE I. Population of periodic orbits of the Lorenz attractor $R=28$.

Period	Number	x coordinate	z coordinate	Sequence
1	1	14.252 206 52	39.786 724 55	1
2	1	14.795 176 95	40.922 164 87	10
3	2	15.037 808 56	41.429 922 32	101
		15.252 574 19	41.879 328 17	100
4	3	14.901 405 34	41.144 554 28	1011
		15.533 322 80	42.467 400 90	1001
		15.632 521 79	42.675 284 35	1000

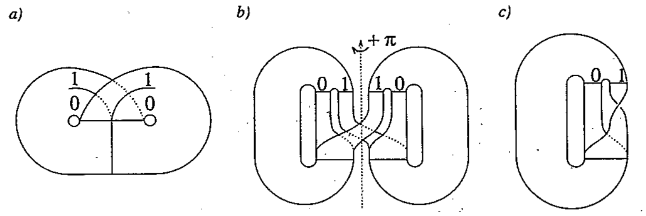

FIG. 2. Extraction of the template from the Lorenz mask: (a) The Lorenz mask. (b) Four band template obtained by division of each wing in two bands: one, labeled 0, associated with the reinjection of the trajectory in the same wing, and one, labeled 1, associated with the transition from one wing to the other. (c) Template of the fundamental domain (one wing): the left band 1 is reinjected in the left wing by a rotation by $+\pi$.

Following the location tree procedure described in [2], N_1 and N_2 are constructed on the template (Fig. 3). Then the linking number $L(N_1,N_2)$ is evaluated by counting the number of signed crossings σ_{ij} using the crossing convention given in [2], according to

$$L(N_i,N_j)=\frac{1}{2}\sum_p \sigma_{ij}(p) \quad (i\neq j)\ , \tag{6}$$

in which p designates a crossing between N_i and N_j. We then obtain $L(101,10)=2$. The physical periodic orbits are then projected on a plane (Fig. 4) and the linking number is again evaluated on these projections. While doing so, it must be remembered that orbits develop on the two-wing attractor, while, conversely, the equivariance implies that one must only consider a one-wing fundamental domain [4]. In particular, when dealing with asymmetric orbits, the pair of asymmetric orbits must be projected on the plane. This is so because such orbits develop on two wings and, therefore, to recover all the relevant information on one wing, the pair must be considered. As a result, the linking number then obtained from the projection must be twice the ones obtained from the template, as we indeed checked.

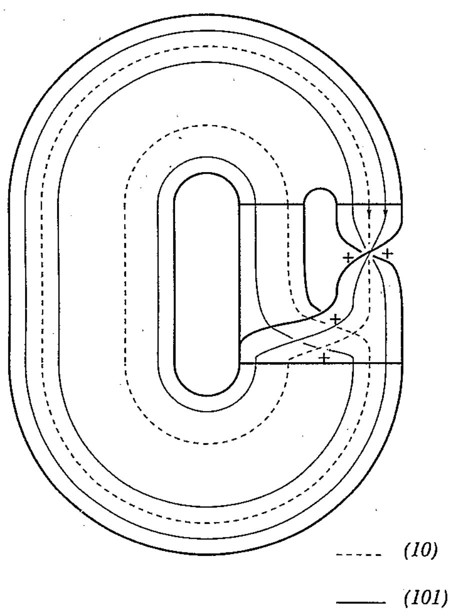

FIG. 3. Orbit pair (101,10) constructed on the template. The linking number $L(101,10)$ is equal to $\frac{1}{2}(+4)=2$.

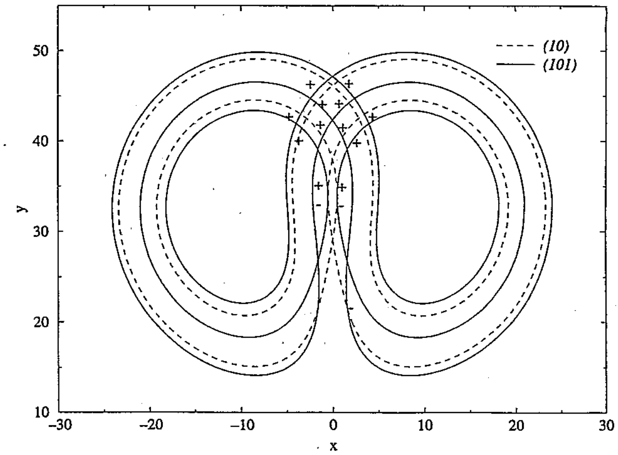

FIG. 4. The orbit pair (101,10) projected on a plane. The linking number $L(101,10)$ is equal to $\frac{1}{2}[\frac{1}{2}(+8)]=2$.

III. THREE LETTER SYMBOLIC DYNAMICS

A. First-return map and periodic orbits

When R increases, the chaos becomes more developed, i.e., a new monotonic branch appears on the first-return map. To avoid artifact splitting of the first two branches due to the development of the stable manifold out of the wing plane [8], the first-return map is constructed by using the variable $w=|x|+2.8z$, which is formed from an invariant variable z, the absolute value of an equivariant variable x [9], and 2.8 is an empirical factor. The map (Fig. 5) displays three branches labeled 0,1,2. The population of periodic orbits (available upon request) is then obtained. There is no sequence (000), i.e., the symbolic dynamics is pruned.

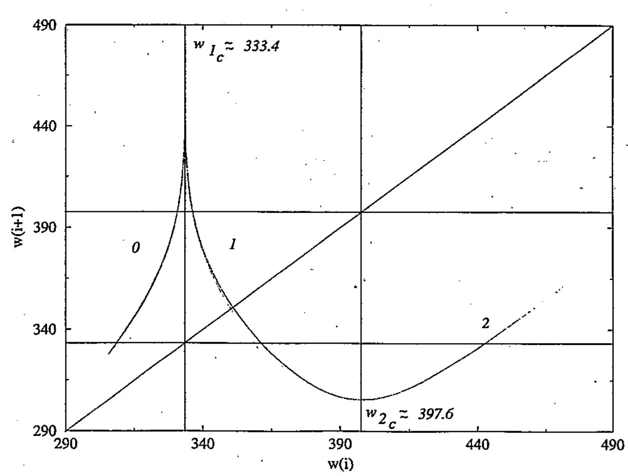

FIG. 5. First-return map from the Poincaré set P_y to itself ($R=90$): $w(i+1)$ versus $w(i)$ where $w(i)=|x(i)|+2.8z(i)$.

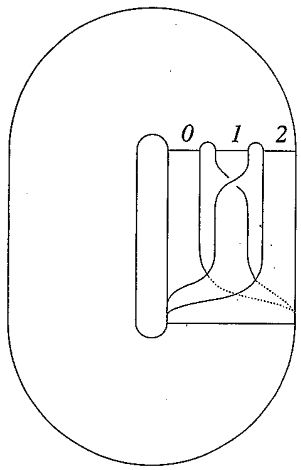

FIG. 6. The Lorenz template for $R = 90$.

B. Template and checking

Starting from the six band Lorenz mask for $(R,\sigma,b)=(90,10,\frac{8}{3})$ and processing similarly as for Fig. 2, we obtain the template in Fig. 6 in which the even local torsion of band 2 is in agreement with the fact that branch 2 is orientation preserving. The template matrix then reads

$$M_L=\begin{bmatrix} 0 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

As in Sec. II E, this template has been successfully checked by comparing linking numbers evaluated from the template and linking numbers evaluated by using orbits projection for the pair (21,10).

IV. CONCLUSION

By using a topological description accounting for the Lorenz vector field equivariance, all periodic orbits up to a three letter symbolic dynamics may be encoded. This requires the use of Poincaré sets generalizing Poincaré sections for equivariant systems. Generated templates are in agreement with the properties of first-return maps. Proposed templates are successfully checked by invoking the values of linking numbers.

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