# Algebraic evaluation of linking numbers of unstable periodic orbits in chaotic attractors

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An algebraic expression for evaluation of linking numbers of unstable periodic orbits in chaotic attractors is demonstrated. An illustrating example (horseshoe dynamics) is provided.

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Achieving a topological description of chaotic attractors is an important objective of nonlinear dynamics. In the past few years several works have tackled this topic. In particular, the idea has arisen that an attractor can be described by the population of its periodic orbits, their related symbolic dynamics, and their linking numbers [1]. The main concept in the topological theory of low-dimensional chaos is the template introduced in the context of hyperbolic flows by Birman and Williams [2]. The template is a branched surface associated with a semiflow whose periodic orbits have the same topological organization as the original flow.

In addition to the geometric view of a template, Mindlin *et al.* [1] describe braid templates by three pieces of algebraic data. The first one is a set of $k$ integers, each one given by the winding number (in multiples of $\pi$) of each branch: they represent the respective local torsion of the $k$ branches. These integers are signed according to the standard crossing convention [3] illustrated in Fig. 1. The second one is a braid word describing the crossing structure of the $k$ branches of the template. The third piece of data is the layering information, which gives the order in which branches are connected to the branch line.

Melvin and Tufillaro [3] reduce this algebraic characterization of a $k$-branch template to $k\times k$ template matrix by introducing a standard insertion convention: branches are ordered back to front from left to right. The linking matrix $M$ is then defined by

$$M\equiv\begin{cases}M(i,i)=(\text{the sum of signed half-twists in the } i\text{th branch})\\ M(i,j)=(\text{the sum of the oriented crossings between the } i\text{th and } j\text{th branches})\quad (i\neq j)\ .\end{cases} \tag{1}$$

A branch of a template is currently called a strand. Each strand is associated with a letter of symbolic dynamics by using a first-return map from a Poincaré section to itself. Therefore each band corresponds to a monotonic branch of the first-return map. A band is characterized by its local torsion whose parity is in agreement with the slope of its corresponding monotonic branch: a band with an even (odd) local torsion corresponds to an increasing (decreasing) monotonic branch. Therefore a band associated with an even (odd) letter is called orientation preserving (reversing) [4]. In this Brief Report we establish a relation to obtain linking numbers from the template matrix and symbolic dynamics.

We recall that the linking number $L(K_1,K_2)$ between two knots (periodic orbits) $K_1$ and $K_2$ is equal to the half-sum of the oriented crossings between $K_1$ and $K_2$ in a regular projection of the link $(K_1,K_2)$ (a drawing of it such that no more than two lines cross at any point) [5]. Define a number $\epsilon(p)=\pm 1$ according to the crossing convention for each crossing $p$ (Fig. 1). Then the linking number $L(K_1,K_2)$ is defined by

$$L(K_1,K_2)=\frac{1}{2}\sum_p \epsilon(p)\ . \tag{2}$$

In like manner, the self-linking number of a periodic orbit $K_1$ is defined as the linking number of the orbit with a thin strip of its stable (or equivalently unstable) manifold [5].

Let us now carry out a review of the different origins of crossings between two periodic orbits of a template. Let $K_1$ and $K_2$ be two orbits of respective periods $p_1$ and $p_2$. Their associated symbolic sequences are therefore of the forms $(\sigma_1,\sigma_2,\ldots,\sigma_{p_1})$ and $(\tau_1,\tau_2,\ldots,\tau_{p_2})$, respectively, where $\sigma_i$ and $\tau_j$ are letters of the symbolic dynamics. Let $B_k$ and $B_l$ be two bands associated with two letters $k$ and $l$ of the symbolic dynamics, respectively. Each band

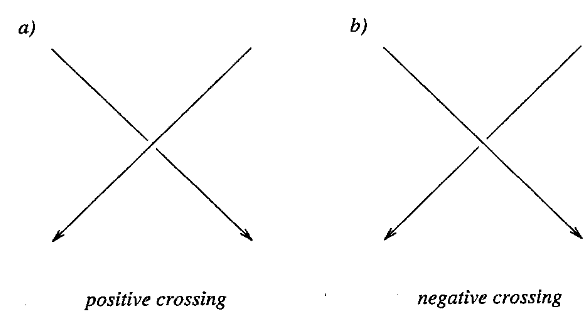


FIG. 1 Standard crossing convention.

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has a local torsion equal to $M(k,k)$ and $M(l,l)$, respectively, and they present $M(k,l)$ crossings on a plane projection of the ribbon graph.

The linking number $L(K_1,K_2)$ is equal to the half-sum of oriented crossings due to (i) local torsion of strands, (ii) crossings between strands, and (iii) crossings in the layering graph.

(i) Local torsions: We suppose that there exists an integer $i \in [1,p_1]$ and an integer $j \in [1,p_2]$ such that $\sigma_i = \tau_j = k$. Orbits $K_1$ and $K_2$ are then passing through strand $B_k$ and present $M(k,k)$ crossings associated with the couple $(\sigma_i,\tau_j)$ on a plane projection of strand $B_k$.

(ii) Strand crossings: We suppose that there exists an integer $i \in [1,p_1]$ and an integer $j \in [1,p_2]$ such that $\sigma_i = k$ and $\tau_j = l$. Therefore when orbits $K_1$ and $K_2$ are passing through strands $B_k$ and $B_l$, respectively, they present $M(k,l)$ crossings associated with the couple $(\sigma_i,\tau_j)$ on a plane projection of the ribbon graph (not accounting for the layering graph).

(iii) Layering graph: Oriented crossings in the layering graph between orbits $K_1$ and $K_2$ may be counted by constructing a layering braid. In order to do that, each periodic point of orbits $K_1$ and $K_2$ must be located on the lower and upper level lines of the braid.

First, periodic points of $K_1$ and $K_2$ are encoded by symbolic sequences which are cyclic permutations of the sequences $(\sigma_1,\sigma_2,\ldots,\sigma_{p_1})$ and $(\tau_1,\tau_2,\ldots,\tau_{p_2})$, respectively. For instance, the $i$th periodic point of $K_1$ is associated with the sequence $(\sigma_1,\sigma_{i+1},\ldots,\sigma_{p_1},\sigma_1,\ldots,\sigma_{i-1})$. Periodic point symbolic sequences are then ordered with respect to the natural order ($k < l$ for instance) [6]. This first arrangement permits one to obtain the lower level line of the layering braid. The upper level line is built up from the naturally ordered periodic points on the lower line taking into account local torsions and strand crossings. In order to do that, two rules are to be applied.

*Rule* 1. If a strand $B_k$ has an odd torsion $M(k,k)$, the order of all periodic points $(\sigma_i,\sigma_{i+1},\ldots,\sigma_{p_1},\sigma_1,\ldots,\sigma_{i-1})$ and $(\tau_j,\tau_{j+1},\ldots,\tau_{p_2},\tau_1,\ldots,\tau_{j-1})$ such that $\sigma_i = \tau_j = k$ is reversed. Let us note that, as a result of the natural ordering, all periodic points $(\sigma_i = \tau_j = k)$ are gathered together on the lower line.

*Rule* 2. Let $S_k$ be the set of periodic points $(\sigma_i,\sigma_{i+1},\ldots,\sigma_{p_1},\sigma_1,\ldots,\sigma_{i-1})$ and $(\tau_j,\tau_{j+1},\ldots,\tau_{p_2},\tau_1,\ldots,\tau_{j-1})$ such that $\sigma_i = k$ and $\tau_j = k$ for $i \in [1,p_1]$ and $j \in [1,p_2]$, respectively. Let $S_l$ be the set of periodic points $(\sigma_i,\sigma_{i+1},\ldots,\sigma_{p_1},\sigma_1,\ldots,\sigma_{i-1})$ and $(\tau_j,\tau_{j+1},\ldots,\tau_{p_2},\tau_1,\ldots,\tau_{j-1})$ such that $\sigma_i = l$ and $\tau_j = l$ for $i \in [1,p_1]$ and $j \in [1,p_2]$, respectively, with $k \neq l$. If the sum of oriented crossings $M(k,l)$ between strands $B_k$ and $B_l$ is odd, sets $S_k$ and $S_l$ are permutated.

The layering braid is afterward obtained by joining each upper periodic point to its cyclic permutation iterate on the lower level line. The layering crossing number $N_{\text{lay}}(K_1,K_2)$ is then evaluated by counting the crossings between strings associated with orbits $K_1$ and $K_2$, respectively. Due to the standard insertion convention, oriented crossings on the layering graph are positive. The linking number is then given by the following relation:

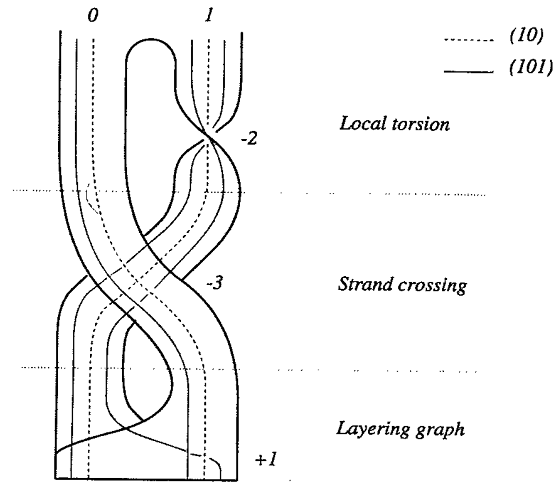


FIG. 2. Template construction of periodic orbits respectively encoded by (10) and (101): the linking number $L(10,101) = -2$.

$$L(K_1,K_2) = \frac{1}{2}\left[\sum_{i=l}^{p_1}\sum_{j=1}^{p_2} M(\sigma_i,\tau_j) + N_{\text{lay}}(K_1,K_2)\right]. \quad (3)$$

In like manner, the self-linking number is then given by relation (3) with $K_2 = K_1$ and $p_1 = p_2$. Thus the layering crossing number $N_{\text{lay}}(K_1,K_1)$ is equal to the sum of crossings between strings of the periodic orbit $K_1$ on the layering braid constructed following the two rules and a copy of these strings which represent the thin strip of the unstable manifold.

*Example*

Let us choose a horseshoe map with the natural order $0 < 1$. The increasing monotonic branch is labeled by 0 and the decreasing one by 1. The chosen linking matrix reads

$$M = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}. \quad (4)$$

Let us now calculate the linking number of two periodic orbits encoded by (10) and (101). With respect to the natural order as defined by Lin [6], we obtain the following arrangement of periodic points:

$$(011) < (01) < (110) < (10) < (101). \quad (5)$$

Periodic orbits are then constructed on the template (Fig. 2).

The layering braid is constructed according to rules 1 and 2:

(i) The lower level line of the layering braid is given by the natural ordering (5):

$$\underbrace{(011)\,(01)}_{S_0} \quad \underbrace{(110)\,(10)\,(101)}_{S_1}$$

exhibiting two sets $S_0$ and $S_1$ associated with strands $B_0$ and $B_1$, respectively.

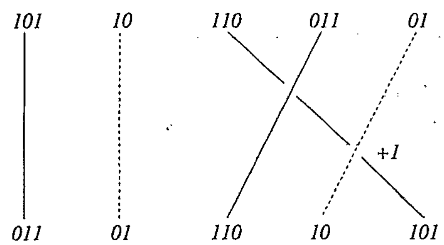


FIG. 3. Layering braid of the link (10, 101).

(ii) Applying rule 1, only $S_1$ is reversed since $M(0,0)$ is even and $M(1,1)$ is odd, leading to

$$\underbrace{(011)\,(01)}_{S_0}\quad\underbrace{(101)\,(10)\,(110)}_{S_1}$$

in agreement with the local torsion graph (Fig. 2).

(iii) Applying rule 2, sets $S_0$ and $S_1$ are permutated since $M(0,1)$ is odd, leading to

$$\underbrace{(101)\,(10)\,(110)}_{S_1}\quad\underbrace{(011)\,(01)}_{S_0}$$

in agreement with the strand crossing graph. This arrangement forms the upper level line of the layering braid (Fig. 3).

(iv) The layering braid is constructed by joining upper periodic points to their respective cyclic permutation iterates on the lower level line processing from left to right (Fig. 3). The constructed layering braid is topologically equivalent to the layering graph of the template (Fig. 2).

The linking number $L(10,101)$ then reads

$$\begin{aligned}L(10,101)&=\tfrac{1}{2}[2M(1,1)+3M(1,0)+M(0,0)\\&\quad+N_{\text{ins}}(10,101)]\\&=\tfrac{1}{2}[-2-3+0+1]\\&=-2\,. \qquad (6)\end{aligned}$$

The linking numbers obtained from relation (3) are in agreement with those determined by counting the oriented crossings between (10) and (101) on the template. This relation appears to offer theoretical and computational advantages to obtain linking numbers from templates and symbolic dynamics.

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